Well-posedness and mathematical analysis of linear evolution equations with a new parameter

by

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Preface

This study was carried out in the Department of Mathematical Sciences, University of South Africa (UNISA), South Africa, Florida, from April 2018 to January 2020, under the supervision of Professor E. F. Doungmo Goufo. This study is the original work of the researcher and has not been submitted in any form, in order to obtain any qualification at any tertiary institution. Where use has been made of works by other authors, they have been duly acknowledged.

Declaration-Plagiarism

Student number: 3440-115-6

I, Victor Tebogo Monyayi, declare that ‘Well-posedness and mathematical analysis of linear evolution equations with a new parameter’ is my own work and that all the sources that I have used or quoted have been indicated and acknowledged by means of complete references.

I further declare that I have not previously submitted this work, or part of it, for examination at the University of South Africa for another qualification or at any other higher education institution.

January 2020

SIGNATURE
(Mr Victor T. Monyayi)

DATE

Dedication

I dedicate this dissertation to the memories of my mother, Rebotile Christinah Monyayi and my grandfather Elias Monyayi.

Acknowledgements

I wish to say thank God, the Almighty Father, for the guidance and for seeing me through this journey. A special thought to the memory of my mother, Rebotile Christinah Monyayi and my grandfather Elias Monyayi, whom I believe have become my guardian angels. I also want to express my sincere gratitude to my supervisor, Prof Emile Franc DOUNGMO GOUFO, for introducing me to the subject of my dissertation, fruitful discussions, constructive criticisms, continuous encouragement and support that I needed to complete this work, without his guidance this dissertation would not have been possible. Many thanks to my beloved sisters, Ntebaleng Annet Motaung, Alettah Mashoto Monyayi and Patricia Ntombizodwa Selahle for their love, prayers, support and continuous encouragement. Finally, thank you to the University of South Africa for giving me a chance to complete this study here.

Abstract

In this dissertation we apply linear evolution equations to the Newtonian derivative, Caputo time fractional derivative and ϖ-time fractional derivative. It is notable that the most utilized fractional order derivatives for modelling true life challenges are Riemann-Liouville and Caputo fractional derivatives, however these fractional derivatives have the same weakness of not satisfying the chain rule, which is one of the most important elements of the match asymptotic method [2, 3, 16]. Furthermore the classical bounded perturbation theorem associated with Riemann-Liouville and Caputo fractional derivatives has confirmed not to be in general truthful for these models, particularly for solution operators of evolution systems of a derivative with fractional parameter φ that is less than one $(0 < \varphi < 1)$ [29]. To solve this problem, we introduce the derivative with new parameter, which is defined as a local derivative but has a fractional order called ϖ-derivative and apply this derivative to linear evolution equation and to support what we have done in the theory, we utilize application to population dynamics and we provide the numerical simulations for particular cases.

Key terms: Mittag-Leffler functions; linear evolution equations; bounded linear operators; Caputo time fractional derivative; ϖ-fractional derivative; perturbation; two-parameter solution operators; well-posedness; population model.

Contents

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List of Acronyms

ACP Abstract Cauchy Problem

BFD $\varpi-$ fractional derivative

BVP Boundary value problem

CFD Caputo fractional derivative

CFFD New Caputo-Fabrizio derivative with fractional order

DC Differential calculus

DE Differential equation

EE Evolution equation

FC Fractional calculus

FD Fractional derivative

FDE Fractional differential equation

FI Fractional integral

IACP Inhomogeneous Abstract Cauchy Problem

IC Initial condition

IVP Initial value problem

LEE Linear evolution equation

MLF Mittag- Leffler function

NACP Non-autonomous Abstract Cauchy Problem

ODE Ordinary differential equation

PDE Partial differential equation

RLFD Riemann-Liouville fractional derivative

Chapter 1

Introduction

Differential calculus (DC) is a subfield of calculus in the field of mathematics that deals with the rates at which quantities change. Differentiation can be defined as the procedure of determining the derivative of a given function [3]. According to Abdon Atangana in [3] today‘s development of calculus is attributed to Gottfried Leibniz (1646-1716) and Isaac Newton (1643-1727), for introducing self-governing and unified methods of differentiation and derivatives. The derivative introduced by Isaac Newton (1643-1727) is known as the Newtonian derivative, also called a normal derivative and its definition is as follows:

Definition 1.0.1 ([3])**.** *Suppose that y is a function specified in $[a, b]$, therefore the first order derivative of $y(x)$ is specified by*

$$\frac{dy}{dx}(x) = y'(x) = \lim_{h\to 0^+} \frac{y(x+h) - y(x)}{h}, \tag{1.0.1}$$

where $[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}$ is a closed interval.

Suppose that y is differentiable on an open interval (a, b), then we have the following information associated with the normal derivative of y

- If $\frac{dy}{dx}(x) < 0$ for all $x \in (a, b)$, then y is a decreasing on (a, b).
- If $\frac{dy}{dx}(x) > 0$ for all $x \in (a, b)$, then y is an increasing on (a, b).
- If $\frac{dy}{dx}(x) = 0$ for all $x \in (a, b)$, then y is constant on (a, b) and the value of x is called a critical point of y.

Newton used the notation $x, \dot{x}, \ddot{x}, \ldots$ and Leibniz used the notation $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots \frac{d^ny}{dx^n}$, where n is a positive integer number. Differentiation is one of the most significant concepts and is utilized to create mathematical models of various life challenges; however, the Newtonian derivative was improved due to the diversity of the physical challenges faced in today's life [3].

The improvement of the Newtonian derivative resulted in the construction of the fractional calculus (FC). Fractional calculus (FC) refers to the derivatives and integrals of arbitrary (fractional) order, that unifies and generalize the ideas of integer-order differentiation and n-fold integration [8]. The beginning of fractional calculus (FC) was proposed in 1695 when L 'Hôspital questioned Leibniz as to what will $\frac{d^n y}{d^n x}$ mean given that n is a fractional value, i.e $n = \frac{1}{2}$. According to [21] Leibnitz 's response was predictive since Leibnitz said this will result to a contradiction, a contradiction which in future valuable outcomes will be found from it since there are few contradictions with no usefulness. Leibnitz 's answer was relevant as we see today that differential equations (DEs) with fractional order turned to be helpful and dominant tools for defining nonlinear phenomena that are associated with numerous parts of science such as biology, engineering, ecology, chemistry and various areas of applied science [3]. Most mathematical models, containing those in continuous time random walk, mathematical epidemiology, biomedical engineering, control theory, Levy statistics, porous media, acoustic dissipation, image processing and fractional signal, fractional Brownian, fractional phase-locked loops, fractional filters motion and non-local phenomena have demonstrated to offer an improved explanation of the phenomenon investigated than models with the conventional integer-order derivative [7, 16]. There are various types of fractional derivatives. Not so long ago, a new Caputo-Fabrizio fractional derivative (CFFD) also known as fractional derivative without singular kernel was developed by Caputo and Fabrizio [6]. This new developed fractional derivative is still under investigation. The most generally utilized fractional derivatives (FD) are Riemann-Liouville fractional derivative (RLFD) and Caputo fractional derivative (CFD). These concepts are utilized for modelling physical problems. However, these derivatives don't fulfil essential properties of the Newtonian concept, for example the chain rule which is considered as the important elements of the match asymptotic method [2, 3, 16]. In [3] the match asymptotic method is specified as the common approach to obtain an accurate approximation to the solution to the system of equations. The conformable fractional derivative was proposed recently [2, 3, 16]. This fractional derivative is theoretically simpler to deal with, furthermore complies with some conventional properties that can't be fulfilled by other existing fractional derivatives, for example, it fulfils the chain rule.

Anyway this fractional derivative has an exceptionally huge shortcoming, which is the fractional derivative (FD) of any function which is differentiable at the point zero gives zero and this doesn't fulfil any physical problem or can't have any physical interpretation [3]. An improved version was developed with the intention to extend the limitation of the conformable derivative, nevertheless this derivative relies on the interval on which the function is being differentiated which is another challenge for some physical problems [3]. To solve this problem, Abdon Atangana and Emile Franc Doungmo Goufo in [2, 3] introduced a derivative with a new parameter called ϖ-fractional derivative or simply Beta-derivative.

The objective of BFD is to additionally expand the match asymptotic method in the scope of the fractional differential equation (FDE), to be utilized to additionally describe the boundary layers problems within the folder of fractional calculus (FC) [2, 3]. Then again, this beta-derivative fulfils each of the 16 criteria of an operator to be called a fractional derivative (FD) [3]. Hence in this dissertation we apply ϖ-derivative to linear evolution equation. More details on this new derivative are given in Chapter 2 and Chapter 4.

1.1 Preliminaries

We give the definitions of Euler gamma function, Mittag- Leffler function (MLF), Laplace transform, convolution and properties which are useful in fractional calculus (FC).

1.1.1 Gamma functions

In [8] $\Gamma(\Phi)$ is called the gamma function and specified as follow:

$$\Gamma(\Phi) = \int_0^\infty e^{-t} t^{\Phi-1} dt \quad Re(\Phi) > 0. \tag{1.1.1}$$

where $t^{\Phi-1} = e^{(\Phi-1)\ln t}$

For $\Phi = n = 1, 2, 3...$ we can show utilizing (1.1.1) that :

$$\Gamma(1) = 1! = 1,$$

$$\begin{aligned}
\Gamma(2) &= 1.\Gamma(1) = 1.1 = 1!, \\
\Gamma(3) &= 2.\Gamma(2) = 2.1 = 2!, \\
\Gamma(4) &= 3.\Gamma(3) = 3.2 = 3!, \\
\cdots & \quad \cdots \\
\Gamma(n+1) &= n\Gamma(n) = n(n-1)! = n!, \\
\Gamma(n) &= (n-1)!
\end{aligned}$$

Since (1.1.1) is related to the factorial by the identity $\Gamma(n+1) = n!$, we can write the formula for the binomial coefficient in terms of the gamma function Γ as follow:

$$^{n}C_{b} = \left(\begin{array}{c} n \\ b \end{array} \right) = \frac{n!}{b!(n-b)!} = \frac{\Gamma(n+1)}{\Gamma(b+1)\Gamma(n-b+1)} \tag{1.1.2}$$

where n and b are positive integers with $n > b$.

Another useful property of the $\Gamma(\Phi)$ is that it fulfils the following functional equation [8]:

$$\Gamma(\Phi+1) = \Phi\Gamma(\Phi),$$

we prove this using integrating by parts:

$$\Gamma(\Phi+1) = \int_0^\infty e^{-t}t^{\Phi}dt = [-e^{-t}t^{\Phi}]_0^\infty + \Phi\int_0^\infty e^{-t}t^{\Phi-1}dt = \Phi\Gamma(\Phi). \tag{1.1.3}$$

We now show that: $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

By definition (1.1.1)

we have

$$\Gamma(\frac{1}{2}) = \int_0^\infty e^{-t}t^{-\frac{1}{2}}dt$$

If we put $t = y^2$, therefore $dt = 2ydy$, and we obtain

$$\Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-y^2}dy \tag{1.1.4}$$

Equivalently we can write

$$\Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-x^2}dx \tag{1.1.5}$$

If we multiply (1.1.4) and (1.1.5) we obtain

$$\left(\Gamma(\frac{1}{2})\right)^2 = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dxdy \tag{1.1.6}$$

Equation (1.1.6) is the double integral on the 1^{st} quadrant and can be calculated in polar coordinates to get

$$\left(\Gamma(\frac{1}{2})\right)^2 = 4\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2}rdrd\theta = \pi$$

Therefore, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

1.1.2 Mittag-Leffler function (MLF)

The MLF for any $z \in \mathbb{C}$ with parameter φ is defined in [8] as:

$$E_\varphi(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\varphi k + 1)} \qquad \mathrm{Re}(z) > 0. \tag{1.1.7}$$

Notice that if $\varphi = 1$, we have $E_1(z) = e^z$.

In [8], the MLF of two parameters $E_{\varphi,\varpi}(z)$ which is a generalization of (1.1.7) is defined by

$$E_{\varphi,\varpi}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\varphi k + \varpi)}, \quad z, \varpi \in \mathbb{C}; \quad \mathrm{Re}(z) > 0. \tag{1.1.8}$$

In [31], the MLF of two parameters is expressed in terms of integral as follows:

$$E_{\varphi,\varpi}(z) = \frac{1}{2\pi i}\int_C \frac{t^{\varphi-\varpi}e^t}{t^\varphi - z}dt, \tag{1.1.9}$$

the contour C begins and ends at $-\infty$ and circle around the disc counter clockwise [31].

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Some of the interesting properties are [8, 20]

$E_{1,1}(z) = e^z$

$E_{2,1}(z)^2 = coshz$

$E_{2,2}(z^2) = \frac{sinhz}{z}$

$E_{\varphi,1}(z) = E_{\varphi}(z)$

$E_{1/2,1}(z) = e^{z^2} erfc(-z)$,

where $erfc(z)$ is called the complementary error function.

1.1.3 Laplace transform and convolution

Laplace transform it was named after its designer Pierre-Simon Laplace and is specified as an integral transformation of a function that transforms the ODEs in to algebraic form so that it becomes easy to be solved and it has numerous applications in science and engineering. The definition of Laplace transform is specified as follow:

Definition 1.1.1 ([20]). *The Laplace transform of the function $f(t)$ is specified by*

$$\mathcal{L}[f(t); s] = \int_0^\infty e^{-st} f(t) dt = F(s), \quad s \in \mathbb{C} \tag{1.1.10}$$

The Laplace transform of the function $f(t) = t^\varphi$ is given for φ as non-integer order $n - 1 < \varphi \leq n$

$$\mathcal{L}[t^n; s] = \frac{\Gamma(\varphi + 1)}{s^{n+1}} = \frac{n!}{s^{n+1}}, \; n = 0, 1, 2, \ldots$$

and it can be shown that

$\mathcal{L}[t^{1/2}; s] = \frac{1}{2}\left(\frac{\pi}{s^3}\right)^{1/2}$,

and

$\mathcal{L}[t^{-1/2}; s] = \left(\frac{\pi}{s}\right)^{1/2}$.

According to [8], the Laplace transform of the nth derivative of $f(t)$ is specified by

$$\mathcal{L}[f^{(n)}(t); s] = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0) = s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-k-1)}(0). \tag{1.1.11}$$

Convolution is an operation on two functions f and g that yields a third function articulating how the shape of one is modified by the other. The convolution of the functions f and g, defined in $[0, \infty)$ is denoted by $f * g$ and is specified by [20]

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\, d\tau = (g * f)(t)$$

Consequently

$$\mathcal{L}(f * g)(t) = F(s)G(s)$$

1.2 Breakdown of the dissertation

Recall that the main objective of this dissertation is to utilize the derivative with a new parameter (ϖ-derivative) to prove that the perturbations by bounded linear operators for linear evolution systems are valid for the derivative with fractional parameter ϖ that is less than one ($0 < \varpi < 1$).

Hence, Chapter 2 is all about literature review on fractional derivatives (FDs) and evolution equations (EEs), we begin with the history of fractional calculus. We also provide some important definitions and properties of fractional integral (FI) and fractional derivatives (FD). Few advantages and disadvantages of fractional derivatives (FDs) are provided. Methods of evaluating the ordinary differential equations and fractional differential equations are also discussed we end the Chapter with literature review on evolution equations.

In Chapter 3, Semi-groups are utilized to apply normal derivative to linear evolution equations, we provide definitions, theorems and apply them to non-autonomous evolution system, and we end the Chapter with the application of old Caputo time fractional derivative applied to linear evolution equations where we introduce the solution operator for fractional differential equations. As it was mentioned on the introduction that with Newtonian derivative (normal derivative) all the complexity of today's life applications can no longer be satisfied and the Caputo fractional derivative does not satisfy the chain rule.

Therefore, in Chapter 4, we show that the ϖ-derivative is a solution to our open problem because is well-possed, that is: the solution exists, the solution is unique and the solution's behaviour changes continuously with the initial conditions, moreover the bounded perturbation theorem is valid when we apply it to linear evolution equation of the form ${}_0^A D_t^{\varpi} u(t) = Au(t)$, $u(0) = f$; $0 < \varpi \leq 1$, $t > 0$ where ϖ is the fractional order, ${}_0^A D_t^{\varpi}$ is a generalized differential operator called beta derivative, u is a suitable function and A is a linear operator in a Banach space. A Banach space Z is a complete normed vector space, which is endowed with the norm $||.||_Z$ and which is complete with respect to the distance function induced by the norm, that is, for every Cauchy sequence $\{z_n\}$ in Z, there exists an element z in Z such that $\lim\limits_{n\to\infty} z_n = z$ or equivalently $\lim\limits_{n\to\infty} ||z_n - z||_Z = 0$.

Last of all, the conclusion and an open problem followed in Chapter 5.

Chapter 2

Literature review

2.1 Introduction

The origin of fractional calculus (FC) was in 1695 when the French mathematician L 'Hôspital (1661-1704) asked the question to the German mathematician and philosopher Gottfried Leibniz (1646-1716) as to the meaning of $\frac{d^k y}{d^k x}$ if $k = 1/2$; Gottfried Leibniz 's answer was that"*this will lead to the paradox from which, one day, useful results will be drawn from it*" [9, 21]. Since then many scientists including Euler, Laplace, Fourier, Abel, Riemann and Liouville were interested in the idea of fractional calculus (FC).

In 1729 Euler took the first step to the right direction by introducing the gamma function $\Gamma(z)$. In 1819, Lacroix became the first to mention the derivative of arbitrary order, but did not go ahead [21].

According to [21], Lacroix started with $y = f(x) = x^r$, where r is a natural number and found the k^{th} derivative to be

$$\frac{d^k f(x)}{d^k x} = \frac{k!}{(r-k)!} x^{r-k}, \quad r \geq k,$$

then he used gamma function Γ to have

$$\frac{d^k f(x)}{d^k x} = \frac{\Gamma(r+1)}{\Gamma(r-k+1)} x^{r-k} \tag{2.1.1}$$

Using this formula when $r = 1$ and $k = \frac{1}{2}$ for the generalized factorial and he obtain

$$\frac{d^{\frac{1}{2}} y}{d^{\frac{1}{2}} x} = \frac{d^{\frac{1}{2}} f(x)}{d^{\frac{1}{2}} x} = \frac{2}{\sqrt{\pi}} x^{\frac{1}{2}}$$

In 1822, Fourier characterized the Fractional derivative (FD) in the form of an integral, he started with the integral representation of $f(x)$ specified by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z)dz \int_{-\infty}^{\infty} cos(px - pz)dp$$

and generalized $f(x)$ as follow (see [21], p. 5)

$$\frac{d^n}{dx^n}f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z)dz \int_{-\infty}^{\infty} p^n cos(px - pz + \frac{n\pi}{2})dp$$

Here n is valid for any value [21], meaning that this could be utilized as a definition of the n^{th} order derivative of a non-integer n.

However, In 1823 Abel was the first to use fractional operations by applying fractional calculus (FC) in the solution of an integral equation that is associated with the tautochrone problem [21]. He solved the following integral equation called Abel integral equation of the 1^{st} kind [9, 21]

$$f(x) = \frac{1}{\Gamma(\varphi)}\int_0^x \frac{\Phi(t)}{(x-t)^{1-\varphi}}dt \quad 0 < \varphi < 1$$

and obtain the following solution

$$\Phi(x) = \frac{1}{\Gamma(1-\varphi)}\frac{d}{dx}\int_0^x \frac{f(t)}{(x-t)^{\varphi}}dt$$

Abel did not actually obtain his solution using fractional calculus but he showed that it could be written as a fractional derivative (FD) and it appears that he was totally not mindful of fractional derivative (FD) or fractional integral (FI) idea. Liouville made few endeavours in 1832, In the main he utilized the exponentials as beginning stage for presenting the fractional derivative (FD). He used them to generalized the standard equation for the derivative of an exponential and applied it to the derivative computation of functions represented by series with exponentials [21]. Grunwald in 1867 and Letnikov in 1868 introduced what is now known as the Grünwald-Letnikov fractional derivative. Their idea was to start with the ordinary derivative.

2.2 Grünwald-Letnikov fractional derivative

In this section we illustrate how Anton Karl Grünwald (1838-1920) from Prague and Aleksey Vasilievich Letnikov from Moscow combined their ideas to develop what so called the Grünwald-Letnikov fractional derivative. This fractional derivative is basically the extension of the normal derivative, they started with the definition of the normal derivative and generalise it, so that it can also take the derivative of non-integer order. Their construction of this fractional derivative was as follows:

Consider the derivative of a function $f(x)$, specified in a closed interval $[a, b]$, therefore the 1^{st} is specified by

$$f'(x) = \frac{df}{dx} = \lim_{h\to 0} \frac{f(x) - f(x-h)}{h}, \tag{2.2.1}$$

The 2^{nd} order derivative gives

$$f''(x) = \frac{d^2 f}{dx^2} = \lim_{h\to 0} \frac{f(x) - 2f(x-h) - f(x-2h)}{h^2}, \tag{2.2.2}$$

the 3^{rd} order derivative gives

$$f'''(x) = \frac{d^3 f}{dx^3} = \lim_{h\to 0} \frac{f(x) - 3f(x-h) + 3f(x-2h) - f(x-3h)}{h^3}, \tag{2.2.3}$$

and , by induction we get

$$f^{(n)}(x) = \frac{d^n f}{dx^n} = \lim_{h\to 0} \frac{1}{h^n} \sum_{m=0}^{n} (-1)^m \left(\begin{array}{c} n \\ m \end{array} \right) f(x - mh), \tag{2.2.4}$$

where

$$\left(\begin{array}{c} n \\ m \end{array} \right) = \frac{n!}{m!(n-m)!} = \frac{\Gamma(n+1)}{\Gamma(m+1)\Gamma(n-m+1)} \tag{2.2.5}$$

Using the generalization of (2.2.4) the derivative for non-integer order can be defined and we get the following definition

Definition 2.2.1 ([8]). *The Grünwald-Letnikov definition of the fractional order differential-integral is specified by*

$$f^{(\varphi)}(x) = \frac{d^\varphi f}{dx^\varphi} = \lim_{h\to 0} \frac{1}{h^\varphi} \sum_{m=0}^{\varphi} (-1)^m \left(\begin{array}{c} \varphi \\ m \end{array} \right) f(x - mh), \quad \varphi \in \mathbb{R} \tag{2.2.6}$$

The Laplace transform of the Grünwald-Letnikov fractional derivative is specified by [[8], p. 107]

$$\mathcal{L}[D^\varphi f(x); s] = s^\varphi F(s), \quad 0 < \varphi < 1$$

2.3 The Riemann-Liouville integral

Let us consider the Cauchy formula for the infinite sequence of n-fold integral [8, 20]

$$J_x^n f(x) := \int_a^x dx_1 \int_a^{x_1} dx_2 \ldots \int_a^{x_{n-1}} f(x_n) dx_n = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t) dt, \tag{2.3.1}$$

holds for $n \in \mathbb{N}$, a, $x \in \mathbb{R}$, $x > a$. If we substitute n by a positive real number φ and generalize $(n-1)!$ using the Gamma function property $\Gamma(n) = (n-1)!$ the following definition is obtained

Definition 2.3.1 ([21]). *Assume that φ, a, $x \in \mathbb{R}$, $\varphi > 0$, $x > a$, therefore the Riemann-Liouville fractional integral with a parameter φ is given by*

$$J_x^{\varphi} f(x) = \frac{1}{\Gamma(\varphi)} \int_a^x (x-t)^{\varphi-1} f(t)dt, \tag{2.3.2}$$

Γ is the gamma function.

We provide the properties of fractional integral which are also found in [8, 20, 24, 25]

(i). By convention

$$J^0 f(x) = f(x)$$

i.e., $J^0 = I$ (identity operator).

(ii). For linearity we have

$$J^{\varphi}(\lambda f(x) + g(x)) = \lambda J^{\varphi} f(x) + J^{\varphi} g(x), \quad \varphi \in \mathbb{R}, \lambda \in \mathbb{C}$$

(iii). If $f(x)$ is continuous for $x \geq 0$ the following hold

$$\lim_{\varphi \to 0} J^{\varphi} f(x) = f(x),$$

$$J^{\varphi}(J^{\varpi} f(x)) = J^{\varpi}(J^{\varphi} f(x)) = J^{\varphi+\varpi} f(x), \quad \varphi > 0, \varpi > 0$$

(iv). The Riemann-Liouville integral of the power function satisfies

$$J^{\varphi} t^{\vartheta} = \frac{\Gamma(\vartheta+1)}{\Gamma(\vartheta+1+\varphi)} t^{\vartheta+\varphi}, \quad \varphi > 0, \vartheta > -1, t > 0,$$

(v). If the fractional derivative with a parameter φ is integrable therefore,

$$J^{\varphi} D_x^{\varphi} f(x) = f(x) - \sum_{k=0}^{n-1} f^k(0) \frac{t^k}{k!}$$

.

The Laplace transform of the fractional integral with a parameter φ (2.3.2) is given by [21]

$$\mathcal{L}[J^{\varphi} f(x); s] = s^{-\varphi} F(s)$$

2.4 The Riemann-Liouville fractional derivative

Let φ, a, $x \in \mathbb{R}$, $\varphi > 0$, $x > a$, therefore the Riemann-Liouville fractional derivative of a function f is specified by [21]:

$$D_x^{\varphi} f(x) = D_x^n J^{n-\varphi} f(x) = \begin{cases} \frac{1}{\Gamma(n-\varphi)} \left(\frac{d}{dx}\right)^n \int_a^x (x-t)^{n-\varphi-1} f(t)dt & for \quad n-1 < \varphi \leq n. \\ \\ \left(\frac{d}{dx}\right)^n f(x) & for \quad \varphi = n, \end{cases} \tag{2.4.1}$$

where $D_x^n = \left(\frac{d}{dx}\right)^n = \frac{d^n}{dx^n} = f^{(n)}$

Some properties are as follow

(i). (2.4.1) is the left-inverse operator of the fractional integral (2.3.2) [21, 25] , i.e.,

$$D^{\varphi} J^{\varphi} = I$$

(ii). By convention

$$D^0 f(x) = f(x)$$

i.e., $D^0 = I$ (identity operator) [21]

(iii). The Riemann-Liouville fractional derivative of the power function fulfils

$$D^{\varphi} t^{\vartheta} = \frac{\Gamma(\vartheta+1)}{\Gamma(\vartheta+1-\varphi)} t^{\vartheta-\varphi}, \quad \varphi > 0, \vartheta > -1, t > 0,$$

which gives the same results as (2.1.1)

(iv). The Riemann-Liouville fractional derivative of a constant K is specified by [21, 25]

$$D^{\varphi}(K) = K \frac{t^{-\varphi}}{(1-\varphi)} \neq 0, \quad t > 0, \varphi > 0$$

The Laplace transform of the Riemann-Liouville fractional differential operator with a parameter φ (2.4.1) is given by [21]

$$\mathcal{L}[D_x^{\varphi} f(x); s] = \begin{cases} s^{\varphi} F(s) & for \quad \varphi > 0 \\ \\ s^{\varphi} F(s) - \sum_{k=0}^{n-1} s^k D_x^{\varphi-k-1}(0) & for \quad \varphi < 0 \end{cases} \tag{2.4.2}$$

where

$n \in \mathbb{N}$, $n-1 < \varphi \leq n$

2.5 The Caputo fractional derivative

Let φ, a, $x \in \mathbb{R}$, $\varphi > 0$, $x > a$, then the Caputo fractional derivative of a function f is specified as [21]

$$ {}^{C}D_x^{\varphi} f(x) = J^{n-\varphi} D_x^n f(x) = \begin{cases} \frac{1}{\Gamma(n-\varphi)} \int_a^x (x-t)^{n-\varphi-1} \left(\frac{d}{dx}\right)^n (f(t))dt & for \quad n-1 < \varphi \leq n \in \mathbb{N}. \\ \left(\frac{d}{dx}\right)^n f(x) & for \quad \varphi = n \in \mathbb{N}, \end{cases} \tag{2.5.1} $$

where $D_x^n = \left(\frac{d}{dx}\right)^n = \frac{d^n}{dx^n} = f^{(n)}$

Some properties are as follow

(i). (2.5.1) is the left-inverse operator of the fractional integral (2.3.2), i.e.,

$$ {}^{C}D^{\varphi} J^{\varphi} = I $$

(ii). By convention

$$ {}^{C}D^{0} f(x) = f(x) $$

i.e., ${}^{C}D^{0} = I$ (identity operator)

(iii). The Caputo fractional derivative of the power function fulfils

$$ {}^{C}D^{\varphi} t^{\vartheta} = \frac{\Gamma(\vartheta+1)}{\Gamma(\vartheta+1-\varphi)} t^{\vartheta-\varphi}, \quad \varphi > 0, \vartheta > -1, t > 0, $$

(iv). The Caputo fractional derivative of a constant K is specified by [21, 22, 25]

$$ {}^{C}D^{\varphi} K = 0 $$

The Laplace transform of the Caputo fractional differential operator with a parameter φ (2.5.1) is given by [21]

$$ \mathcal{L}[{}^{C}D_x^{\varphi} f(x); s] = \begin{cases} s^{\varphi} F(s) & for \quad \varphi > 0 \\ s^{\varphi} F(s) - \sum_{k=0}^{n-1} s^{\varphi-k-1} f^{(k)}(0) & for \quad \varphi < 0 \end{cases} \tag{2.5.2} $$

where
$n \in \mathbb{N}$, $n-1 < \varphi \leq n$

Advantages and disadvantages of RLFD and CFD

The advantage of Riemann-Liouville fractional derivative is that it is not necessarily important for an arbitrary function to be continuous at the starting point and it is not necessary to be differentiable [3]. However, the Caputo fractional derivative is more appropriate than Riemann-Liouville fractional derivative to be utilized for modelling physical problem because with the Caputo fractional derivative the initial and boundary conditions can be utilized when managing real world problems and one of the very important advantages of the Caputo fractional derivative is that it can allow the traditional initial and boundary conditions to be involved in the formulation of the problem [3]. Moreover the Caputo fractional derivative of a constant yields zero, which implies that this type of derivative can be utilized to model the rate of change, while that one of Riemann-Liouville fractional derivative of a constant is not zero. Furthermore the Laplace transform of the Caputo fractional derivative (2.5.2) is a generalization of the Laplace transform of the integer-order derivative (1.1.11), where n is substituted by φ. The same doesn't hold for Laplace transform of the Riemann-Liouville (2.4.2) this is another important advantage of the Caputo fractional derivative when compared to the Riemann-Liouville fractional derivative.

Lemma 2.5.1 ([20]). *Assume that $n \in \mathbb{N}$, $n-1 < \varphi < n$, $\varphi \in \mathbb{R}$, such that ${}^{C}D_x^{\varphi} f(x)$ exists for $f(x)$.*
Then

$$ {}^{C}D_x^{\varphi} f(x) = J^{n-\varphi} D_x^n f(x). \tag{2.5.3}$$

This implies that the Caputo fractional derivative is equivalent to $(n-1)$- fold integration after $n-th$ order differentiation. While the Riemann- Liouville fractional operator is equivalent to $(n-1)$- fold integration after $n-th$ order differentiation, but in reverse order, hence we also have the following results:

Lemma 2.5.2 ([20]). *Assume that $n \in \mathbb{N}$, $n-1 < \varphi < n$, $\varphi \in \mathbb{R}$, such that $D_x^{\varphi} f(x)$ exists for $f(x)$.*
Then

$$D_x^{\varphi} f(x) = D_x^n J^{n-\varphi} f(x). \tag{2.5.4}$$

From (2.5.3) and (2.5.4), since $J^{n-\varphi} D_x^n \neq D_x^n J^{n-\varphi}$ it follows that

Proposition 2.5.3 ([20]). *Caputo fractional operator and the Riemann- Liouville fractional operator, do not coincide, i.e.,*

$$ {}^{C}D_x^{\varphi} f(x) \neq D_x^{\varphi} f(x)$$

Theorem 2.5.4 ([20]). *Assume that $x > 0$, $\varphi \in \mathbb{R}$, $n-1 < \varphi < n \in \mathbb{N}$. therefore the following relation between ${}^{C}D_x^{\varphi} f(x)$ and $D_x^{\varphi} f(x)$ holds*

$$ {}^{C}D_x^{\varphi} f(x) = D_x^{\varphi} f(x) - \sum_{k=0}^{n-1} \frac{x^{k-\varphi}}{\Gamma(k+1-\varphi)} f^{(k)}(0). \tag{2.5.5}$$

Proof. [20, Theorem 2.24] ■

Remark 2.5.5. (2.5.5) *implies that the Caputo and the Riemann-Liouville fractional derivatives coincide if and only if $f(x)$ together with its first $n-1$ derivatives vanish at $t = 0$.*

Corollary 2.5.6 ([14, 20])**.** *There is a relationship between Caputo and the Riemann-Liouville fractional derivative:*

$$ {}^{C}D_x^{\varphi} f(x) = D^{\varphi}\left(f(x) - \sum_{k=0}^{n-1} f^{(k)}(0)\frac{x^k}{k!} \right), \quad n-1 < \varphi \leq n \tag{2.5.6}$$

(2.5.6) Can also be written as follow:

$$ {}^{C}D_x^{\varphi} f(x) = D_x^{n} J_x^{n-\varphi}\left(f(x) - \sum_{k=0}^{n-1} f^{(k)}(0)\frac{x^k}{k!} \right), \quad n-1 < \varphi \leq n$$

where J_x^{φ} is the Riemann-Liouville fractional integral:

2.6 Derivative without singular kernel and other definitions

The new Caputo Fabrizio fractional derivative CFFD, also called the derivative with non-singular kernel is just an extension of the old Caputo fractional derivative where the kernel of the integral has been reformulated. Consider the old Caputo time fractional derivative given in [6] as

$$ {}^{C}D_x^{\varphi} f(x) = \frac{1}{\Gamma(n-\varphi)} \int_a^x \frac{\dot{f}(t)}{(x-t)^{\varphi}} dt$$

where $\varphi \in [0, 1]$, $a \in [-\infty, x]$, $f \in H^1(a, b)$ $b > a$. By substituting the kernel $(x-t)^{-\varphi}$ with the function $exp(-\frac{\varphi x}{1-\varphi})$ and $\frac{1}{\Gamma(n-\varphi)}$ with $\frac{M(\varphi)}{1-\varphi}$, we get new CFFD defined in [6, 7] as

$$ {}^{CFFD}D_x^{\varphi} f(x) = \frac{M(\varphi)}{(1-\varphi)} \int_a^x \dot{f}(t) exp[-\frac{\varphi(x-t)}{1-\varphi}] dt \tag{2.6.1}$$

According to [7], the generalization of the Sobolev space is given by

$$ H^n(a, b) = \{f : f, \frac{d}{dx} f, ..., D_x^n f \in L^2(a, b)\},$$

where $L^2(a, b)$ is the space of square integrable function on (a, b) and $M(\varphi)$ is a normalization constant such that $M(0) = M(\infty) = 1$. In accordance with definition (2.6.1), the new definition of fractional time derivative is zero when $f(x)$ is a constant

If f does not belong to $H^1(a, b)$, the above formula is re-formulated for $f \in L^1(-\infty, b)$

and for $\varphi \in [0,1]$ to read as

$$^{CFFD}D_x^{\varphi}f(x) = \frac{\varphi M(\varphi)}{(1-\varphi)}\int_{-\infty}^{x}(f(x)-f(t))exp[-\frac{(x-t)}{\varphi}]dt \tag{2.6.2}$$

where $f \in L^1(-\infty, b)$ and $M(\varphi)$ is a normalization constant such that $M(0) = M(\infty) = 1$. It is clear that compared to the old Caputo, the kernel does n't have singularity at $x = t$ as is the case with the old Caputo.

In addition the New Caputo-Fabrizio derivative with fractional order (CFFD) fulfils the following relations $\forall f$, where f is a suitable function (see [7]):

$$\lim_{\varphi\to 1} {}^{CFFD}D_x^{\varphi}f(x) = \dot{f}(x)$$

and

$$\lim_{\varphi\to 0} {}^{CFFD}D_x^{\varphi}f(x) = f(x) - f(a),$$

where a is the initial point of the integro-differentiation.

The definition of the CFFD was modified by Losada and Nieto [4] to be

$$^{CFFD}D_x^{\varphi}f(x) = \frac{(2-\varphi)M(\varphi)}{2(1-\varphi)}\int_{0}^{x}\dot{f}(x)exp[-\frac{\varphi(x-t)}{1-\varphi}]dt \tag{2.6.3}$$

The fractional integral of the CFFD was proposed by Losada and Nieto [4] and proved to be:

$$^{CFFD}I_x^{\varphi}f(x) = \frac{2(2-\varphi)}{(1-\varphi)M(\varphi)}f(x) + \frac{2\varphi}{(1-\varphi)M(\varphi)}\int_{a}^{x}f(t)dt \tag{2.6.4}$$

$\varphi \in [0,1], \quad t \geq 0$. This fractional integral of the CFFD is seen as a type of an average between function u and its integral of order one. The Laplace transform of the fractional integral of the CFFD is specified by

$$\mathcal{L}[^{CFFD}D_x^{\varphi}f(x), s] = \frac{s\bar{u}(x,s) - u_0(x)}{s + \varphi(1-s)}$$

here $\bar{u}(x,s)$ is the Laplace transform $\mathcal{L}(u(x,t),s)$ of $u(x,t)$.

There exist in literature few definitions of fractional derivatives, we provide some versions according to the references [1, 2, 3, 5, 23]:

1. The improved Riemann-Liouville fractional derivative of a function u was proposed by Guy Jumarie [3], and is specified as

$$^{L}D_x^{\varphi}u(x) = \frac{1}{\Gamma(n-\varphi)}\left(\frac{d}{dx}\right)^n\int_{0}^{x}(x-t)^{n-\varphi-1}(u(t)-u(0))dt, \quad n-1 < \varphi \leq n. \tag{2.6.5}$$

2. The local fractional derivative of a function u is specified as

$$L_x^{\varphi}(u(x)) = \lim_{x \to x_0} \frac{\Gamma(1+\varphi)(u(x) - u(x_0))}{(x - x_0)^{\varphi}}, \tag{2.6.6}$$

3. The conformable fractional derivative of a function u is specified as

$$T_{\varphi}(u(x)) = \lim_{\varepsilon \to 0} \frac{u\left(x + \varepsilon x^{1-\varphi}\right) - u(x)}{\varepsilon}, \tag{2.6.7}$$

4. The improved conformable fractional derivative of a give function u is defined in the open interval (a, b) is specified as

$$T_{\varphi}(u(x)) = \lim_{\varepsilon \to 0} \frac{u\left(x + \varepsilon (x-a)^{1-\varphi}\right) - u(x)}{\varepsilon}, \tag{2.6.8}$$

(2.6.7) and (2.6.8) appear to fulfil some similar properties of the standard idea of derivative, but as we stated before they have certain shortcoming that will not let them to be utilized in modelling real world problems [2]. Then, in [2, 3, 16] Abdon Atangana and Emile Franc Doungmo Goufo introduced a derivative with new parameter called ϖ-derivative and defined it as follow:

2.7 ϖ-fractional derivative (BFD)

Definition 2.7.1 ([2, 3, 16, 28, 29, 32, 33, 34]). *Suppose that u is a function such that, $u : [a, \infty) \to \mathbb{R}$ and $a \in \mathbb{R}$. Therefore, the BFD of a function u respect to t is specified as follow:*

$$ {}_0^A D_t^{\varpi} u(t) = \begin{cases} \lim\limits_{\varepsilon \to 0} \frac{u\left(t + \varepsilon\left(t + \frac{1}{\Gamma(\varpi)}\right)^{1-\varpi}\right) - u(t)}{\varepsilon} & \forall 0 < \varpi \leq 1, t \geq 0 \\ u(t) & \forall \varpi = 0, t \leq 0 \end{cases} \tag{2.7.1}$$

where Γ denotes the gamma-function

$$\Gamma(\zeta) = \int_0^{\infty} t^{\zeta - 1} e^{-t} dt.$$

A function u is said to be beta-differentiable if the above limit exists. It is also easy to show that for $\varpi = 1$, we obtain ${}_0^A D_t^{\varpi} u(t) = \frac{d}{dt} u(t)$. In addition not like other fractional derivatives, the beta-differentiable can be defined the similar way as the normal derivative, mean that, it can be locally specified at a particular point [2].

We can comment that the definition of ϖ-derivative does n't rely on the interval on which the function is specified the same as in Definition (2.6.8). If the function u is differentiable, the definition at a point zero is not zero, which was not the case in other definitions [2].

For a general order, assume that we have $k\beta$, therefore the $k\beta$-derivative of u is specified as:

$$ {}_0^A D_t^{k\beta} u(t) = {}_0^A D_t^{\varpi} ({}_0^A D_t^{(k-1)\varpi} u(t)) \quad \forall 0 < \varpi \leq 1, \quad t \geq 0, \quad k \in \mathbb{N} $$

Note that $k\beta$-derivative of a given function gives information about the previous $(k-1)\varpi$-derivatives of the similar function. For example, we have:

$$ \begin{aligned} D_t^{2\varpi} u(t) &= D_t^{\varpi} \left(D_t^{\varpi} u(t) \right) \\ &= \left(t + \frac{1}{\Gamma(\varpi)} \right)^{1-\varpi} \left[(1-\varpi) \left(t + \frac{1}{\Gamma(\varpi)} \right)^{-\varpi} u' + \left(t + \frac{1}{\Gamma(\varpi)} \right)^{1-\varpi} u'' \right]. \end{aligned} \tag{2.7.2} $$

This tells us that the BFD has a unique property that other derivative do not have. It is additionally simple to confirm that if $\varpi = 1$, we obtain the 2^{nd} derivative of u.

Theorem 2.7.2. *Suppose that $f : [a, \infty) \to \mathbb{R}$ a given function which is ϖ-differentiable at a point $t_0 \geq a$, $\varpi \in (0, 1]$, therefore f is continuous at x_0.*

Proof. [2, Theorem 2] ■

Theorem 2.7.3. *Assume that f is ϖ-differentiable on an open interval (a, b) then*

- *If ${}_0^A D_t^{\varpi} f(t) < 0$ $\forall t \in (a, b)$ therefore f is decreasing on (a, b);*
- *If ${}_0^A D_t^{\varpi} f(t) > 0$ $\forall t \in (a, b)$ therefore f is increasing on (a, b);*
- *If ${}_0^A D_t^{\varpi} f(t) = 0$ $\forall t \in (a, b)$ therefore f is constant on (a, b);*

Proof. [2, Theorem 3] ■

Theorem 2.7.4. *Assuming that, $g \neq 0$ and f are two functions ϖ-differentiable with $\varpi \in (0, 1]$ therefore, the following relations could be fulfilled*

- ${}_0^A D_t^{\varpi} (af(t) + bg(t)) = a {}_0^A D_t^{\varpi} (f(t)) + b {}_0^A D_t^{\varpi} (g(t))$ $\forall$ $a, b \in \mathbb{R}$;
- ${}_0^A D_t^{\varpi} (c) = 0$ *where c is any constant;*
- ${}_0^A D_t^{\varpi} (f(t)g(t)) = g(t) {}_0^A D_t^{\varpi} (f(t)) + f(t) {}_0^A D_t^{\varpi} (g(t))$;
- ${}_0^A D_t^{\varpi} \frac{(f(t)}{g(t))} = \frac{g(t) {}_0^A D_t^{\varpi} (f(t)) - f(t) {}_0^A D_t^{\varpi} (g(t))}{g^2(t)}$;

Proof. [2, Theorem 4] ■

Theorem 2.7.5. *Assume that $f : [a, \infty) \to \mathbb{R}$ is a given function such that f is both differentiable and ϖ-differentiable. Let g be a function that is also differentiable and specified in the range of f, therefore the following rule applies*

$${}_0^A D_t^{\varpi}(gof(t)) = \left(t + \frac{1}{\Gamma(\varpi)}\right)^{1-\varpi} f'(t)g'(f(t))$$

Proof. [2, Theorem 5] ■

Definition 2.7.6. *Assume that $f : [a, \infty) \to \mathbb{R}$ is a function , therefore the ϖ- fractional integral of f is given by*

$${}_0^A I_t^{\varpi}(f(t)) = \int_a^t \left(x + \frac{1}{\Gamma(\varpi)}\right)^{1-\varpi} f(x)dx \tag{2.7.3}$$

The above integral is the inverse operator of the ϖ-fractional derivative and we support the above definition by the following theorem.

Theorem 2.7.7. ${}_0^A D_t^{\varpi}[{}_0^A I_t^{\varphi} f(t)] = f(t)\ \forall t \geq 0$ *with f a given continuous and differentiable function.*

Proof. [2, Theorem 7] ■

Theorem 2.7.8. ${}_0^A I_t^{\varpi}[{}_0^A D_t^{\varpi} f(t)] = f(t) - f(a)\ \forall t \geq 0$ *with f a given continuous and differentiable function.*

Proof. [2, Theorem 8] ■

2.7.1 Partial derivative with new parameter

Definition 2.7.9 ([3])**.** *Let u be a function of two variable x and y, therefore, the ϖ-derivative of u with respect to x is specified as follow:*

$${}_0^A D_x^{\varpi} u(x, y) = \lim_{\varepsilon \to 0} \frac{u\left(x + \varepsilon\left(t + \frac{1}{\Gamma(\varpi)}\right)^{1-\varpi}, y\right) - u(x, y)}{\varepsilon}, \tag{2.7.4}$$

for $0 < \varpi \leq 1$

The reader may consult [2] for more details and properties on beta derivative.

2.8 Methods of evaluating ODEs and FDEs

There are different methods for evaluating the ODEs and FDEs. Here we shall name only few. The iterative method is a method, which is easy to solve only simple FDEs with arbitrary real order [32]. Laplace transform method, which can be used in both ODEs and FDEs, however in the case of FDEs (normally the CFD or RLFD) is associated with the Mittag-Leffler function which is specified as $E_{\varphi}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\varphi k+1)}$ $\quad \mathrm{Re}(z) > 0$. and $E_{\varphi,\varpi}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\varphi k+\varpi)}, \quad z, \varpi \in \mathbb{C}; \quad \mathrm{Re}(z) > 0$. for one parameter $E_{\varphi}(z)$ and two parameters $E_{\varphi,\varpi}(z)$. The Laplace transform method eliminates some challenges included when dealing with iterative technique, in addition the Laplace transform is a very important tool in many fields of science such as chemistry, physics, engineering and signal processing. In [3] Abdon Atangana applied different techniques associated with a new derivative called beta-fractional derivative such as the Variational iteration method Homotopy decomposition method, Sumudu decomposition method and Laplace decomposition method.

In Chapter 3 we will illustrate how to apply the Laplace transform method associated with FDEs in the sense of the Old Caputo fractional derivative and in Chapter 4 we find the solution of Initial value problem associated with beta derivative using the beta exponential function which is given as

$$\mathcal{E}_{\varpi}(t) = Exp\left[k\left(\frac{\left(t+\frac{1}{\Gamma(\varpi)}\right)^{\varpi} - \Gamma(\varpi)^{-\varpi}}{\varpi}\right)\right]. \tag{2.8.1}$$

However it should be noted that it is possible to determine the existence of a solution, its uniqueness and whether the existing solution is continuously depending on the data without attempting to solve the equation, if these three criteria are fulfilled we say that the Initial value problem (IVP) is well-posed. The well-posedness of our initial value problem will be determined using the semigroup theory.

2.9 Evolution equations

Evolution equation is interpreted as the differential law of the evolution (development) in time of a system. The term doesn't have an accurate definition, and its importance depends on the equation itself, yet in addition on the formulation of the problem for which it is utilized. Such equation gives the potential outcomes to develop solutions from the provided initial condition which its interpretation is described as the initial state of the system. There are different kinds of evolution equations, the first to be introduce was in the form of the ODE. In this section we will only deal with ODE for homogeneous and Inhomogeneous linear evolution equation Linear evolution equation (LEE).

Consider the following (linear) initial value problem IVP

$$\begin{cases} \frac{du(t)}{dt} = Au(t) & \forall t \in (0, \infty) \\ u(0) = z \end{cases} \tag{2.9.1}$$

also known as abstract Cauchy problem. Here $u(t)$ is the solution of (2.9.1) which is the description of the state of the system at time t that changes in time at the rate provided by a linear operator A . In this section and last two Chapters we will show how to apply the theory of semigroup to LEE with respect to different types of the derivatives.

2.9.1 Homogeneous abstract Cauchy problem of autonomous evolution system

A homogeneous system is just a system of linear equations where all constants on the right side of the equals sign are zero. For example the system of differential equations can be written as: $X'(t) = AX(t) + f(t)$. If f(t) is equal to zero: $f(t) = 0$, then the system is said to be homogeneous: $X'(t) = AX(t)$.

Assume that Z is a Banach space and A is a linear operator with $D(A) \subset Z$. If $z \in Z$ therefore, the Abstract Cauchy Problem (ACP) with Initial condition (IC) z is given by the following homogeneous IVP

$$\begin{cases} \frac{du(t)}{dt} = Au(t) & \forall t \in (0, \infty) \\ u(0) = z \end{cases} \tag{2.9.2}$$

here the solution of $u(t)$ refers to an Z valued function $u(t)$. The function $u(t)$ needs to be continuous for $t \in [0, \infty)$, continuously differentiable and $u(t) \in D(A)$ for $t \in (0, \infty)$, then we say (2.9.2) is fulfilled. According to [10] If A is the infinitesimal generator of a C_0 semigroup $V(t)$, then the ACP for A has unique solution, that is $u(t) = V(t)z$, $\forall z \in D(A)$.

Theorem 2.9.1. *Given that A is the infinitesimal generator of the differential semigroup therefore $\forall z \in Z$ the IVP (2.9.2) has a solution which is unique.*

Proof. [10, Theorem 1.4] ∎

2.9.2 Inhomogeneous abstract Cauchy problem of autonomous evolution system

Consider the following inhomogeneous IVP

$$\begin{cases} \frac{du(t)}{dt} = Au(t) + f(t) & \forall t \in (0, \infty) \\ u(0) = z \end{cases} \tag{2.9.3}$$

where $f : [0, V] \to Z$. A is the infinitesimal generator of a C_0 semigroup $V(t)$ so that the homogeneous IVP, (i.e., with $f = 0$) that corresponds to (2.9.2) has the solution that unique $\forall z \in D(A)$.

Definition 2.9.2 ([10]). *A function $u : [0, V] \to Z$ is a solution of (2.9.3) on $[0, V]$ if u is continuous on $[0, V]$, continuously differentiable on $[0, V]$ and $u(t) \in D(A)$ for $t \in (0, V)$ and (2.9.3) is fulfiled on $[0, V]$.*

Assume that $V(T)$ is a C_0 semigroup generated by A and u is a solution of (2.9.3). Therefore the Z valued function $g(\tau) = V(t-\tau)u(\tau)$ is differentiable for $0 < \tau < t$ so that

$$\begin{aligned} \frac{dg}{d\tau} &= -AV(t-\tau)u(\tau) + V(t-\tau)u'(\tau) \\ &= -AV(t-\tau)u(\tau) + V(t-\tau)Au(\tau) + V(t-\tau)f(\tau) \\ &= V(t-\tau)f(\tau) \end{aligned} \tag{2.9.4}$$

If $f \in L^I(0, V : Z)$ therefore, $V(t-\tau)f(\tau)$ is integrable and if we integrate utilizing the interval from 0 to t we obtain

$$u(t) = V(t)z + \int_0^t V(t-\tau)f(\tau)d\tau$$

where $V(t)z$ is the solution of the homogeneous IVP (2.9.2).

Definition 2.9.2 implies that it is possible to utilize the corresponding homogenous IVP to determine the existence of the solution of the inhomogeneous IVP. In the next Chapter we continue with the application of linear evolution applied to ODE and CFD.

Chapter 3

Application of normal derivative and Caputo fractional derivatives to linear evolution systems

3.1 Introduction

Since semigroup is a generalization of exponential functions because their properties are the same. Hence it is very important to begin with the definitions and theorems of the semi-group, which are very useful in the application of linear evolution equations, we show how to apply theory of evolution semi-group on autonomous evolution system to determine the well-posedness, we also consider the inhomogeneous and homogeneous initial value problems of non-autonomous evolution systems. The second part of this Chapter we focus on the application of Caputo time fractional derivatives to linear evolution equations where we will utilize the solution operator to determine the well-posedness.

3.2 Linear semi-groups

Consider the following autonomous homogeneous evolution equation

$$\begin{cases} \frac{du(t)}{dt} = Au(t) \qquad \forall t \in (0, \infty) \\ u(0) = z \qquad A \in Z, \end{cases} \tag{3.2.1}$$

where A is linear, but is not necessary for it to be bounded on Z with the domain $D(A) \subset Z$. If we assume that Z is a Banach space with the norm $||.||$, then we have the following theorems and definitions

Definition 3.2.1 ([10]). *Assume that Z is a Banach space, therefore a family $\{V(t)\}_{t\geq 0}$ of a bounded linear operator on Z is said to be a semigroup of bounded linear operator on Z if*

(i) $V(0) = I$
(ii) $V(t+s) = V(t)V(s)$ $\forall t, s \geq 0$ *(semigroup property).*
If , in addition, $\forall z \in Z$, $V(t)z \to z$ *as* $t \to 0$, *then the semigroup is called* C_0*-semigroup or strongly continuous semigroup.*

Definition 3.2.2. *An infitesimal generator* A *is defined by*

$$Az = \lim_{t \to 0^+} \frac{V(t)z - z}{t}$$

and the domain of the above definition is

$$D(A) = \{z \in Z : \lim_{t \to 0^+} \frac{V(t)z - z}{t} \quad exists\}$$

is called the infinitesimal generator of the family of semigroup $\{V(t)\}_{t \geq 0}$.

Definition 3.2.3. *A semigroup of bounded linear operator* $V(t)$, *is said to be uniformly continuous if*

$$\lim_{t \to 0^+} ||V(t) - I|| = 0.$$

Definition 3.2.4. *A semigroup of bounded linear operator* $V(t)$, *is said to be strongly continuous if*

$$\lim_{t \to 0^+} ||V(t)z - z|| = 0 \quad z \in Z.$$

or equivalently

$$\lim_{t \to 0^+} V(t)z = z \quad \forall z \in Z.$$

Theorem 3.2.5. *Every uniformly continuous semigroup* $\{V(t)\}_{t \geq 0}$ *on a Banach space* Z *is specified by*

$$V(t) = e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}, \quad t \geq 0$$

for some bounded linear operator A.

Theorem 3.2.6. *Let* $\{V(t)\}_{t \geq 0}$ *be a strongly continuous semigroup on a Banach space* Z. *There exist constants* $\omega \geq 0$ *and* $M \geq 1$ *such that*

$$||V(t)|| \leq Me^{\omega t}, \qquad 0 \leq t < \infty.$$

Proof. [10, Theorem 2.2] ∎

Definition 3.2.7. *The semigroup* $V(t)$ *is called a contraction semigroup if* $||V(t)|| \leq 1$ $\forall t \geq 0$.

Definition 3.2.8. *The resolvent set $\rho(A)$ of the operator A is given by*

$$\rho(A) = \{\lambda \in \mathbb{C} : R(\lambda; A) \in \mathcal{B}(Z)\},$$

where $R(\lambda; A) = (\lambda I - A)^{-1}$ is called the resolvent operator associated with the operator A.

Theorem 3.2.9 ([10])**.** *(Hille-Yosida Theorem). A linear operator A is the infinitesimal generator of a C_0 semigroup of contractions $V(t)$, $t \geq 0$ if and only if*

(i) A is closed and $D(A) = Z$.
(ii) The resolvent set $\rho(A)$ of A contains $\mathbb{R}$ and $\forall \lambda > 0$,

$$||R(\lambda; A)|| \leq \frac{1}{\lambda}.$$

.

Proof. [10, Theorem 3.1] ■

Theorem 3.2.10. *The abstract Cauchy problem (3.2.1) is said to be well-posed if and only if A is the generator of a semigroup $V(t)$ and the unique solution is provided by $u(t) = V(t)z$, $\forall z \in D(A)$ fulfilling.*

$$z \in C([0, \infty); D(A)) \cap C^1([0, \infty); Z).$$

3.3 Inhomogeneous abstract Cauchy problem of non-autonomous evolution system

We begin this section with the definition of inhomogeneous IVP of non-autonomous evolution system, which is specified as follows:

$$\begin{aligned} \frac{du(t)}{dt} &= \mathcal{A}(t)u(t) + f(t) \qquad for \quad s < t \leq V \\ u(s) &= z \end{aligned} \tag{3.3.1}$$

If $f(t)$ is an Z valued function, Z is a Banach space and $A(t) : D(A(t)) \subset Z \to Z$ is a linear operator in Z, $\forall t \in [0, V]$, we can construct the following theorem:

Theorem 3.3.1 ([10])**.** *An Z valued function $u : [s, V] \to Z$ is a classical solution of (3.3.1) if u is continuous on $[s, V]$, $u(t) \in D(A(t))$ for $s < t \leq V$, u is continuously differentiable on $s < t \leq V$ and fulfils (3.3.1) .*

The existence of the solution of the inhomogeneous IVP (i.e., with $f = 0$) is related to the existence of the solution of the homogeneous IVP via the formula of variations of constants [10], therefore we will concentrate more on the homogeneous initial value problems (IVP).

3.4 Homogeneous abstract Cauchy problem of non-autonomous evolution system

Consider the following homogeneous initial value problem

$$\begin{aligned} \frac{du(t)}{dt} &= A(t)u(t) \qquad for \quad 0 \leq s < t \leq V \\ u(s) &= z \end{aligned} \tag{3.4.1}$$

where $A(t)$ is a bounded linear operator on Z and $t \to A(t)$ is continuous in the uniform operator topology, therefore we have the following theorem:

Theorem 3.4.1 ([10, 19, 30])**.** *Assume that Z is a Banach space, $\forall t \in [0, V]$ and $A(t)$ is a bounded linear operator on Z. if the function $t \to A(t)$ is continuous in the uniform operator topology therefore $\forall z \in Z$ the abstract Cauchy problem (3.4.1) has the unique classical solution u given by the relation:*

$$u(t) = z + \int_s^t A(\zeta)u(\zeta)d(\zeta). \tag{3.4.2}$$

Proof. [10, Theorem 5.1] ■

We define the solution operator of the abstract Cauchy problem (3.4.1) by
$U(t, s)z = u(t) \qquad \forall s, t \in [0, V]$
where u is the solution of (3.4.1) and $U(t, s)$ is a two parameter family of operator. If $A(t) = A$ does not dependent on t therefore $U(t, s) = U(t - s)$ and the two parameter is reduced to one parameter family $U(t)$, $\forall t \in [0, \infty)$, which is the semi-group generated by A. We have the following:

Theorem 3.4.2 ([10])**.** *) $\forall s, t \in [0, V]$, $U(t, s)$ is a bounded linear operator and*
(a) $\|U(t, s)\| \leq exp(\int_s^t \|A(\tau)\| d\tau)$.
(b) $U(t, t) = I$, $U(t, s) = U(t, r)U(r, s)$ $\forall s, r, t \in [0, V]$
(c) $(t, s) \to U(t, s)$ is continuous in the uniform operator topology $\forall s, t \in [0, V]$
(d) $\partial U(t, s)/\partial t = A(t)U(t, s)$ $\forall s, t \in [0, V]$
(e) $\partial U(t, s)/\partial s = -U(t, s)A(s)$ $\forall s, t \in [0, V]$.

Proof. [10, Theorem 5.2] ■

For more details about linear semigroups the reader can consult [10, 11, 19, 30].

3.5 Application of Caputo time fractional derivatives to linear evolution equations

In this section we demonstrate the Laplace transform method and semigroups are applied to the Old Caputo frational initial value problem and perturbation classical results are discussed.

3.5.1 Evaluation of the Caputo initial value problem utilizing Laplace transform method

Let $\varphi > 0$, $t > 0$ and $n \in \mathbf{N}$ be such that $n-1 < \varphi \leq n$. Then consider the initial value problem (IVP):

$$D_t^{\varphi} u(t) = Au(t), \quad u^{(k)}(0) = x_k, \quad k = 0, ..., n-1 \tag{3.5.1}$$

Theorem 3.5.1. *The solution of problem (3.5.1) is given by*

$$u(t) = \sum_{k=0}^{n-1} x_k t^k E_{\varphi,k+1}(At^{\varphi}), \tag{3.5.2}$$

where $E_{\varphi,\varpi}(z)$ is the two- parameter function of Mittag-Leffler type.

Proof. Applying the Laplace transform to the fractional differential equation in (3.5.1) it becomes

$$s^{\varphi} U(s) - \sum_{k=0}^{n-1} s^{\varphi-k-1} u^{(k)}(0) = AU(s) \tag{3.5.3}$$

where $U(s)$ is the Laplace transform of $u(t)$ and since Laplace transform is linear. Equation (3.5.3) can be solved with respect to $U(s)$ as follows

$$U(s) = \sum_{k=0}^{n-1} \frac{s^{\varphi-k-1}}{s^{\varphi} - A} u^{(k)}(0) \tag{3.5.4}$$

Substituting the initial conditions from (3.5.1) into (3.5.4) it follows

$$U(s) = \sum_{k=0}^{n-1} \frac{s^{\varphi-k-1}}{s^{\varphi} - A} x_k \tag{3.5.5}$$

According to [20], p. 42, utilizing the Laplace transform of the two-parameter function of Mittag-Leffler type as well as the linearity property we get

$$U(s) = \sum_{k=0}^{n-1} \frac{s^{\varphi-k-1}}{s^{\varphi} - A} x_k = \sum_{k=0}^{n-1} \mathcal{L}\{t^k E_{\varphi,k+1}(At^{\varphi}); s\} x_k = \mathcal{L}\{\sum_{k=0}^{n-1} x_k t^k E_{\varphi,k+1}(At^{\varphi}); s\} \tag{3.5.6}$$

Then using the inverse Laplace transform, $u(t)$ can be found as

$$u(t) = u(t, \varphi) = \sum_{k=0}^{n-1} x_k t^k E_{\varphi,k+1}(At^{\varphi}),$$

and that completes the proof ■

3.5.2 Initial value problem for fractional evolution equation

Consider a linear closed operator A densely in a Banach space X. Let $\varphi > 0$ and $n \in \mathbf{N}$ be such that $n-1 < \varphi \leq n$. Given that $x \in X$, we investigate the following Cauchy problem for the fractional evolution of order φ:

$$D_t^{\varphi} u(t) = Au(t), \quad u(0) = x, \quad u^{(k)}(0) = 0, \quad k = 1, ..., n-1 \tag{3.5.7}$$

where D_t^{φ} is the Caputo fractional derivative of order φ and the connection between D_t^{φ} and the Riemann-Liouville fractional derivative yields (see[12, 13, 14, 15, 26, 27]):

$$D_t^{\varphi} u(t) = D_t^n J_t^{n-\varphi} \left(u(t) - \sum_{k=0}^{n-1} u^{(k)}(0) \frac{t^k}{k!} \right), \quad n-1 < \varphi \leq n$$

where J_t^{ϖ} is the Riemann-Liouville fractional integral:

$$J_t^{\varpi} u(t) = \frac{1}{\Gamma(\varpi)} \int_0^t (t-s)^{\varpi-1} u(s) ds, \quad \beta > 0, \quad t > 0; \quad J_t^0 u(t) = u(t)$$

Similarly to the ordinary differentiation and integral, for function f such that the following operations are well defined, we have

$$D_t^{\varphi} J_t^{\varphi} f(t) = u(t); \quad J_t^{\varphi} D_t^{\varphi} f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0) \frac{t^k}{k!} \quad n-1 < \varphi \leq n$$

Therefor the Cauchy problem (3.5.7) is equivalent to the following Volterra integral equation

$$u(t) = x + \frac{1}{\Gamma(\varphi)} \int_0^t (t-s)^{\varphi-1} Au(s) ds$$

In the next section we define solution operator of (3.5.7)

3.5.3 Preliminary results

Let X be a Banach space and let A denote a closed linear operator in X with the domain $D(A)$, the resolvent set $\rho(A)$ of the operator A and $R(\lambda; A) = (\lambda I - A)^{-1}$ is the resolvent operator associated with the operator A. Let $\mathcal{B}(X)$ be the space of all bounded linear operators in X, the we have the following:

Definition 3.5.2. *A family $\{S_\varphi(t)\}_{t\geq 0} \subset \mathcal{B}(X)\}$ is called a solution operator for the Cauchy problem (3.5.7) if the following conditions are fulfilled:*
(i) $S_\varphi(t)$ is strongly continuous for $t \geq 0$ and $S_\varphi(0) = I$;
(ii)$S_\varphi(t)D(A) \subset D(A)$ and $AS_\varphi(t)x = S_\varphi(t)Ax$ $\forall x \in D(A)$, $t \geq 0$;
(iii) $S_\varphi(t)x$ is a solution of (3.5.7) $\forall x \in D(A)$, $t \geq 0$;

[12, 15]), The Cauchy problem (3.5.7) is well -posed if it admits a solution operator. furthermore, if the Cauchy problem (3.5.7) has a solution operator $S_\varphi(t)$, then the corresponding problem with initial condition in general form $u^{(k)}(0) = x_k, \quad k = 0, ..., n-1$ is uniquely solvable with solution

$$u(t) = \sum_{k=0}^{n-1} J_t^k S_\varphi(t) x_k$$

provided that $x_k \in D(A)$, $k = 0, ..., n-1$. Therefore, it is well -posed. For this reason, we restrict ourselves from to the problem (3.5.7).

Definition 3.5.3. *The solution operator $S_\varphi(t)$ is said to be exponentially bounded, if there exist the constants $\omega \geq 0$ and $M \geq 1$ such that*

$$||S_\varphi(t)|| \leq Me^{\omega t}, \qquad t \geq 0 \tag{3.5.8}$$

An operator A is said to belong to $\mathcal{C}^\varphi(M, \omega)$ if the Cauchy problem (3.5.7) has a solution operator $S_\varphi(t)$ that satisfy (3.5.8). Denote $\mathcal{C}^\varphi(\omega) = \bigcup\{\mathcal{C}^\varphi(M, \omega); M \geq 1\}$, $\mathcal{C}^\varphi = \bigcup\{\mathcal{C}^\varphi(\omega); \omega \geq 1\}$. In these notations $\mathcal{C}^1$ and $\mathcal{C}^2$ are the sets of all infinitesimal generators of C_0-semigroups and cosine families, respectively.

If $A \in \mathcal{C}^\varphi(M, \omega)$ and $\Re e\lambda > \omega$, then $\lambda^\varphi \in \rho(A)$ and

$$\lambda^{\varphi-1} R(\lambda^\varphi, A) = \int_0^\infty e^{-\lambda s} S_\varphi(s) ds$$

The following generation theorem is a particular case of the results

Theorem 3.5.4 ([12, 14])**.** *Let $\varphi > 0$. Then $A \in \mathcal{C}^\varphi(M, \omega)$ if $(\omega^\varphi, \infty) \subset \rho(A)$ and*

$$||\frac{\partial^n}{\partial \lambda^n}(\lambda^{\varphi-1} R(\lambda^\varphi, A))|| \leq \frac{Mn!}{(\lambda - \omega)^{n+1}}, \quad \lambda > \omega \quad , n = 0, 1, 2... \tag{3.5.9}$$

For more details and similar results the reader can refer to [12, 13, 14, 15].

3.5.4 Perturbation classical results

If A is the generator of a C_0-semigroup and $B \in \mathcal{B}(X)$, then $A + B$ is again a generator of a C_0-semigroup. This is not true in general for solution operator of (3.5.7) with $0 < \varphi < 1$. We show that this results are true by the following example.

Example 3.5.1. *Consider the order of the derivative φ fixed in the interval $(0, 1)$ and the Banach space $H = l^1$ the space of infinite sequence $u = \{u_n\}_{n=1}^{\infty}$ with $u_n \in \mathbb{C}$, with the norm $||u|| = \sum_{n=1}^{\infty} |u_n| < \infty$. Let A_φ be an operator given by $A_\varphi u = \{exp(\frac{i\alpha\pi}{2})nu_n\}_{n=1}^{\infty}$ with the domain $D(A_\varphi) = \{u \in l^1 : \sum_{n=1}^{\infty} n|u_n| < \infty\}$. Then A_φ is the generator of a solution operator of the (3.5.7), but there is no solution operator of (3.5.7) that is generated by the sum $A_\varphi + B$ where $B = I$.*

Proof. For the proof of this example, see [[14], Example 3.1].

Since the Caputo time fractional derivative doesn't fulfil the chain rule and the bounded perturbation theorem has demonstrated not to be all in all valid for these models, particularly for the models associated with the Caputo fractional evolution with the fractional order that is below one $(0 < \varphi < 1)$. Hence, in the next Chapter we utilize the new fractional derivative called ϖ-derivative to prove the well-posedness and to demonstrate that the bounded perturbation is valid for ϖ- fractional evolution with the fractional order that is below one. Moreover this new derivative is defined as a local derivative but has a fractional order called ϖ-derivative.

Chapter 4

Application of beta derivative to linear evolution equation

4.1 Introduction

As it was stated that the well known classical definition of the derivative (definition of the normal derivative) can no longer fulfil the complexities of today's life when coming to model some physical problems. Hence fractional derivatives was introduced in order to solve this problem, however it was found that the most popular derivatives such as Riemann-Liouville fractional derivative (RLFD) and Caputo fractional derivative (CFD) also have major shortcoming of not obeying the chain and quotient rules, which is another problem, furthermore the classical bounded theorem were proven not to be true when applying these fractional derivatives to the linear evolution equation associated with the fractional parameter φ that is less than one, which is another problem. Hence, in this Chapter we introduce a new fractional derivative which is relatively new in the literature called $\varpi-$ fractional derivative (BFD). This fractional derivative was defined in chapter 2 literature review of this dissertation and it was also defined in [3, 16, 29, 32, 33, 34]. Very important advantage of the beta fractional derivative is that it can be defined the same way as the local derivative at the certain point and it satisfies the chain rule and all other properties that are needed to be fulfilled in order for a derivative to be called a fractional derivative. Hence we will investigate the well-posedness of the (ϖ) -derivative applied to linear evolution equation of the form ${}_0^A D_t^{\varpi} u(t) = Au(t)$, $u(0) = f$; $0 < \varpi \leq 1$, $t > 0$ where ${}_0^A D_t^{\varpi}$ is the generalized differential operator called ϖ -fractional derivative, ϖ is a fractional order and A is a closed densely defined operator in a Banach space. Abdon Atangana in [3] introduced the beta exponential function which we shall use to find the solution for evolution system associated with the beta derivative unlike in the case of Riemann-Liouville fractional derivative (RLFD) and Caputo fractional derivative (CFD) where we utilized the Mittag-Leffler function and the Laplace transform to find the solution. The theory of semigroup will be utilized to prove the well-posedness of the above fractional derivative. We also show by the aid of perturbation that evolution system associated with BFD is valid for $0 < \varpi < 1$ We also cosider the application to population dynamics and numerical simulations for some particular cases are performed.

4.2 Solution using beta exponential function

We recall some important notions that have been fully detailed and analyzed in [16, 29]. Considering the following model expressed in terms of the ϖ–derivative:

$$\begin{cases} D_t^{\varpi} u(t) = Au(t), \quad 0 < \varpi \leq 1, \;\; t > 0. \\ u(0) = x \end{cases} \tag{4.2.1}$$

Our main motivation followed from the fact that the system

$$\begin{cases} D_t^{\varpi} u(t) = ku(t), \quad 0 < \varpi \leq 1, \;\; t > 0, \; k \in \mathbb{C} \\ u(0) = x_0. \end{cases} \tag{4.2.2}$$

The following exponential function is know as beta-exponential function and it was introduced by Abdon Atangana in [3]

$$\mathcal{E}_{\varpi}(t) = Exp\left[k\left(\frac{\left(t + \frac{1}{\Gamma(\varpi)}\right)^{\varpi} - \Gamma(\varpi)^{-\varpi}}{\varpi}\right)\right]. \tag{4.2.3}$$

Therefore the solution of the system (4.2.2) yields

$$u(t) = x_0 \mathcal{E}_{\varpi}(t) = x_0 Exp\left[k\left(\frac{\left(t + \frac{1}{\Gamma(\varpi)}\right)^{\varpi} - \Gamma(\varpi)^{-\varpi}}{\varpi}\right)\right]. \tag{4.2.4}$$

It is easy to deduce that if $\varpi = 1$, we get following:

$$u(t) = x_0 e^{kt}.$$

which is known as classical result.

4.3 Two-parameter solution operators associated with beta derivative

Now if we consider the following model

$$\begin{cases} {}_0^A D_t^{\varpi} u(t) = Au(t), \quad 0 < \varpi \leq 1, \;\; t > 0. \\ u(0) = x \end{cases} \tag{4.3.1}$$

Suppose that $A : H \to H$ is a bounded linear operator therefore, the following result applies

Theorem 4.3.1. *For the system (4.3.1), every uniformly continuous two-parameter solution operator $(\{S_\varpi(t)\}_{t\geq 0,\, 0<\varpi\leq 1}; T_\varpi(t))$ on a Banach space H leads to a solution of form:*

$$u(t) = G(T_\varpi)x = \exp\left[A\left(\frac{\left(t+\frac{1}{\Gamma(\varpi)}\right)^{\varpi} - \Gamma(\varpi)^{-\varpi}}{\varpi}\right)\right]x, \qquad x \in D(A),$$

where A is a linear operator which is bounded.

Proof. [16, Theorem 3.1] ■

Definition 4.3.2. *Assume that $\varpi \in (0,1]$ and $t \in \mathbb{R}_+$. Therefore $(\{S_\varpi(t)\}_{t\geq 0,\, 0<\varpi\leq 1}; T_\varpi(t))$ is called a strongly continuous two-parameter solution operator for the equation (4.2.1), if the two-parameter family $\{S_\varpi(t)\}_{t\geq 0,\, 0<\varpi\leq 1}$ is such that*

- *$S_\varpi(t) = G_A(T_\varpi)$ for all times T_ϖ, $t > 0$ where*

$$G_A(T_\varpi)x = \exp\left[A\left(\frac{\left(t+\frac{1}{\Gamma(\varpi)}\right)^{\varpi} - \Gamma(\varpi)^{-\varpi}}{\varpi}\right)\right]x, \qquad x \in D(A),$$

and

T_ϖ is the modified time named the revamped time and is expressed as

$$T_\varpi = T_\varpi(t) = \frac{\left(t+\frac{1}{\Gamma(\varpi)}\right)^{\varpi} - \Gamma(\varpi)^{-\varpi}}{\varpi}, \tag{4.3.2}$$

- *$\{G_A(T_\varpi)\}_{T_\varpi\geq 0}$ should be a one-family strongly continuous semigroup (in T_ϖ) created by the linear operator A, which should also fulfil*

(a) $G_A(0) = I$,

(b) $G_A(T_\varpi + S_\varpi) = G_A(T_\varpi)G_A(S_\varpi)$ *for all* $T_\varpi, S_\varpi \geq 0$;

(c) $\lim_{T_\varpi\to 0^+} G_A(T_\varpi)x = x$ *for any* $x \in H$.

Remark 4.3.3. *It is noticeable that:*

(i) If $\varpi = 1$ and $T_\varpi(t) = t$, then the definition above will coincides with the definition of the classical well known C_0-semigroup of one-parameter.

(ii) If $(\{S_\varpi(t)\}_{t\geq 0,\, 0<\varpi\leq 1}; T_\varpi(t))$ is a strongly continuous two-parameter solution operator for the equation (4.2.1) generated by A, therefore we have the following,

$$Ax = \lim_{t\to 0^+}\frac{S_\varpi(t)x - x}{t} = \lim_{T_\varpi\to 0^+}\frac{G_A(T_\varpi)x - x}{T_\varpi}, \tag{4.3.3}$$

whose domain of the above $D(A)$, is chosen to be specified as the set of all $x \in H$

for which this limit exists.

The equality of (4.3.3) is caused by the fact that $T_\varpi(t) \to 0$ as $t \to 0$. and the above definition.

(iii) *If $(\{S_\varpi(t)\}_{t\geq 0, 0<\varpi\leq 1}; T_\varpi(t))$ is a strongly continuous two-parameter solution operator for the eqution (4.2.1) generated by A, therefore, $\forall x \in D(A)$ the function $t \longrightarrow S_\varpi(t)x = G_A(T_\varpi)x$ is a classical solution of the fractional Cauchy problem (4.2.1).*

(iv) *If $(\{S_\varpi(t)\}_{t\geq 0, 0<\varpi\leq 1}; T_\varpi(t))$ is a strongly continuous two-parameter solution operator for the equation (4.2.1) generated by A, therefore, $\forall x \in D(A)$ the function $T_\varpi \longrightarrow G_A(T_\varpi)x$ a classical solution of*

$$\begin{cases} \partial_t u(t) = Au(t), \quad t > 0. \\ u(0) = x \end{cases} \tag{4.3.4}$$

Proposition 4.3.4. *Let $(\{S_\varpi(t)\}_{t\geq 0, 0<\varpi\leq 1}; T_\varpi(t))$ be the a strongly continuous two-parameter solution operator for the equation (4.2.1) generated by $(A, D(A))$. Therefore $t \to S_\varpi(t)x = G_A(T_\varpi)x, \quad x \in D(A)$, is the only solution of (4.2.1) allowing the values in $D(A)$.*

Proof. See [16, Proposition 3.2] ∎

We will need the following definition for two-parameter solution operators ϖ-exponentially bounded:

Definition 4.3.5.
- *The strongly continuous two-parameter solution operator $(\{S_\varpi(t)\}_{t\geq 0, 0<\varpi\leq 1}; T_\varpi(t))$ for the equation (4.2.1) is called ϖ-exponentially bounded if there exist two constants $\omega \geq 0$ and $M \geq 1$ such that*

$$\|S_\varpi(t)\| \leq MExp\left[\omega\left(\frac{\left(t + \frac{1}{\Gamma(\varpi)}\right)^\varpi - \Gamma(\varpi)^{-\varpi}}{\varpi}\right)\right] \tag{4.3.5}$$

- *If the system (4.2.1) admits a strongly continuous two-parameter solution operator $(\{S_\varpi(t)\}_{t\geq 0, 0<\varpi\leq 1}; T_\varpi(t))$ fulfilling (4.3.5), therefore we say that the operator $A \in \mathcal{G}^\varpi(M, \omega)$.*
- *$(\{S_\varpi(t)\}_{t\geq 0, 0<\varpi\leq 1}; T_\varpi(t))$ is called a contractive if $\|S_\varpi(t)\| \leq 1$, and we say $A \in \mathcal{G}^\varpi(1, 0)$.*
- *As in [18], the problem (4.2.1) is well-posed if it admits a strongly continuous two-parameter solution operator.*

Remark 4.3.6. *The condition (4.3.5) holds if and only if the one parameter family $\{G_A(T_\varpi)\}_{T_\varpi \geq 0}$ given in the Definition 4.3.2 satisfies*

$$\|G_A(T_\varpi)\|_H \leq Me^{\omega T_\varpi} \tag{4.3.6}$$

4.4 Bounded perturbations for two-parameter solution operators

As we illustrated in Chapter 3, example 3.5.1, that the bounded perturbation theorem is not true for models of type (4.2.1) using other fractional derivative like Caputo. In this section we prove that bounded perturbation theorem is valid for the model (4.2.1) with the ϖ–derivative. We consider the following classical bounded perturbation problem:

Problem 4.4.1. *Let $(A, D(A))$ be the above operator in (4.2.1) such that $D(A) \subseteq H$ and $A \to H$ is the generator of a strongly continuous two-parameter solution operator. Consider a second operator $B : D(B) \subseteq H \to H$. Obtain conditions such that the addition of $A + B$ generates a strongly continuous two-parameter solution operator.*

Note that the sum $A + B$ is specified as $(A + B)z := Az + Bz$ for $z \in D(A + B) := D(A) \cap D(B)$. We set $\mathcal{B}(H) := \mathcal{B}(H; H)$ the space of all bounded linear operators from H to H.

Theorem 4.4.1. *(**Bounded perturbations**) Let H be a Banach space and $(A, D(A))$ the generator of a ϖ-exponentially bounded strongly continuous two-parameter solution operator $(\{S_\varpi(t)\}_{t\geq 0,\, 0<\varpi\leq 1}; T_\varpi(t))$ for the equation (4.2.1). If $B \in \mathcal{B}(H)$ therefore, the operator sum $A + B$ is also the generator of a ϖ-exponentially bounded strongly continuous two-parameter solution operator on H.*

Proof. $(\{S_\varpi(t)\}_{t\geq 0,\, 0<\varpi\leq 1}; T_\varpi(t))$ is ϖ-exponentially bounded then, $\exists\, \omega \geq 0$ and $M \geq 1$ such that

$$\|S_\varpi(t)\| \leq MExp\left[\omega\left(\frac{\left(t + \frac{1}{\Gamma(\varpi)}\right)^{\varpi} - \Gamma(\varpi)^{-\varpi}}{\varpi}\right)\right] \tag{4.4.1}$$

According to the Definition, 4.3.2 we also have $\|G_A(T_\varpi)\| \leq Me^{\omega T_\varpi}$. Since $\{G_A(T_\varpi)\}_{T_\varpi\geq 0}$ is a one-family strongly continuous semigroup (in T_ϖ) generated by the operator A, then, making use of [10, Theorem 1.1, Chap.3], there exist a strongly continuous semigroup

$$\{G_{A+B}(T_\varpi)\}_{T_\varpi\geq 0}$$

generated by the sum $A + B$ on H satisfying

$$\|G_{A+B}(T_\varpi)\| \leq Me^{(\omega+M\|B\|)T_\varpi}.$$

Thus, using the Remark 4.3.6, for the model

$$\begin{cases} D_t^{\varpi}u(t) = (A + B)u(t), & 0 < \varpi \leq 1, \;\; t > 0. \\ u(0) = x\ , \end{cases} \tag{4.4.2}$$

there is a strongly continuous two-parameter solution operator, say $(\{Q_\varpi(t)\}_{t\geq 0,\, 0<\varpi\leq 1}; T_\varpi(t))$ such that

$$\{G_{A+B}(T_\varpi)\}_{T_\varpi\geq 0} = \{Q_\varpi(t)\}_{t\geq 0,\, 0<\varpi\leq 1}$$

and satisfying

$$\|Q_\varpi(t)\| \leq MExp\left[(\omega + M\|B\|)\left(\frac{\left(t+\frac{1}{\Gamma(\varpi)}\right)^\varpi - \Gamma(\varpi)^{-\varpi}}{\varpi}\right)\right].$$

Hence, $(\{Q_\varpi(t)\}_{t\geq 0,\, 0<\varpi\leq 1}; T_\varpi(t))$ is ϖ-exponentially bounded and the theorem is proved. ■

Next we provide a relation between the ϖ-exponentially bounded strongly continuous two-parameter solution operator $(\{S_\varpi(t)\}_{t\geq 0,\, 0<\varpi\leq 1}; T_\varpi(t))$ generated by A and $(\{Q_\varpi(t)\}_{t\geq 0,\, 0<\varpi\leq 1}; T_\varpi(t))$ generated by $A+B$.

Corollary 4.4.2. *Consider two strongly continuous two-parameter solution operators* $(\{Q_\varpi(t)\}_{t\geq 0,\, 0<\varpi\leq 1}; T_\varpi(t))$ *with generator* A *and* $(\{Q_\varpi(t)\}_{t\geq 0,\, 0<\varpi\leq 1}; T_\varpi(t))$ *with generator* $A+B$ *on the Banach space* H, *where* $B \in \mathcal{B}(H)$. *Then,*

$$\|S_\varpi(t) - Q_\varpi(t)\| \leq$$
$$M\left(Exp\left[\omega\left(\frac{\left(t+\frac{1}{\Gamma(\varpi)}\right)^\varpi - \Gamma(\varpi)^{-\varpi}}{\varpi}\right)\right]\right)\left(Exp\left[M\omega\|B\|\left(\frac{\left(t+\frac{1}{\Gamma(\varpi)}\right)^\varpi - \Gamma(\varpi)^{-\varpi}}{\varpi}\right)\right]-1\right) \tag{4.4.3}$$

Proof. The proof follows from the previous theorem and the fact that, for every $x \in D(A) = D(A+B)$ we have the relation

$$G_A(T_\varpi)x = G_{A+B}(T_\varpi)x + \int_0^{T_\varpi} G_{A+B}(T_\varpi - S)BG_A(S)x\,dS$$

leading to

$$\|G_A(T_\varpi)x - G_{A+B}(T_\varpi)x\| \leq Me^{\omega T_\varpi}\left(e^{M\omega\|B\|T_\varpi} - 1\right)\|x\|. \tag{4.4.4}$$

Substituting the revamped time T_ϖ yields the assertion. ■

4.5 Application to population dynamics

In this section we prove that it is possible to apply the framework of two-parameter solution operators to some real life phenomena. We focus on population differential equations where delay is considered and given as

$$D_t^\varpi u(t) = -bu(t) + au(t-\tau). \tag{4.5.1}$$

Here the quantity $u(t)$ represents the number of people of a given population at time t, while b, a and τ are constants so that $0 \leq b, a < \infty$ and $0 < \tau < \infty$ and respectively denote the death rate, the birth rate, and the delay due, for instance, to pregnancy in the population.

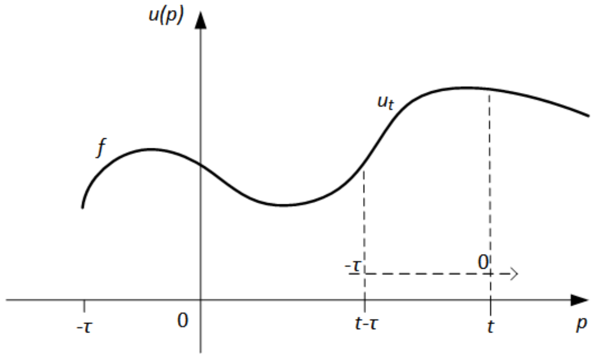

Figure 4.1: Numerical solutions illustrating the time development of the history intervals $[-\tau, 0]$, $\tau > 0$,.

Remark 4.5.1. *Note that, for the sake of population biology interpretation, model* (4.5.1) *is not is not deterministic if we analyze it in the state space $H := \mathbb{C}$ and then, it cannot be possible to use the theory of two-parameter solution operators aforementioned. To bypass this issue, we consider a state space that also includes suitable history information for the system. An example is given by the time evolution of the history intervals* $[-\tau, 0]$, $\tau > 0$, *that reads as*

$$u_t : \ [-\tau, 0] \longrightarrow \ H$$

$$\tau \ \longmapsto u_t(\tau) = u(t + \tau) \tag{4.5.2}$$

and depicted in Fig. 1.

We cannot proceed with the analysis of (4.5.1) without considering some initial conditions. Hence, a function that represents the system's prehistory should be the suitable one here and reads as

$$f : \ [-\tau, 0] \longrightarrow \ H$$

$$t \ \longmapsto f(t) = u(t). \tag{4.5.3}$$

From (4.5.1), (4.5.2) and (4.5.3) we can adopt the following settings: Let $\mathcal{H}$ be a Banach space equipped with the standard sup-norm, we consider its associated space

$$H := C([-\tau, 0], \mathcal{H})$$

that is also a Banach space all continuous functions on $[-\tau, 0]$ that take their values in $\mathcal{H}$. According to [16, Corollary 3.4], the operator $(\mathbb{B}, D(\mathbb{B}))$ where $\mathbb{B}v(t) \equiv -av(t)$ is the generator of a strongly continuous two-parameter solution operator $(\{J_\varpi(t)\}_{t\geq 0,\, 0<\varpi\leq 1}; T_\varpi(t))$ on $\mathcal{H}$. From Definition 4.3.2, we have $J_\varpi(t) = G_{\mathbb{B}}(T_\varpi)$ for all times T_ϖ, $t > 0$ and

$$\{G_{\mathbb{B}}(T_\varpi)\}_{T_\varpi \geq 0} \tag{4.5.4}$$

is a one-family strongly continuous semigroup (in T_ϖ) generated by the operator $\mathbb{B}$ on

$\mathcal{H}$. For the reason of simplicity we denote the delay operator θ by $\theta u_t(\tau) \equiv au(t-\tau)$. This yields the following population abstract delay differential model

$$D_t^{\varpi} u(t) = \mathbb{B}u(t) + \theta u_t, \quad t > 0, \tag{4.5.5}$$

assumed to satisfy the initial condition

$$u(0) = f. \tag{4.5.6}$$

with $f \in H$. To show existence of a ϖ-exponentially bounded strongly continuous two-parameter solution operator for the model (4.5.5)-(4.5.6), we introduce the corresponding population delay differential operator $(A, D(A))$ on H given by

$$Ag := D_t^{\varpi} g(t), \tag{4.5.7}$$

$$D(A) := \{g \in C([-\tau, 0], \mathcal{H}) : \ g(0) \in D(\mathbb{B}) \text{ and } D_t^{\varpi} g|_{t=0} = \mathbb{B}g(0) + \theta g\}).$$

We can now state the following results

Proposition 4.5.2. *The differential operator $(A, D(A))$ given by (4.5.7) is the generator of a strongly continuous two-parameter solution operator $(\{S_{\varpi}(t)\}_{t \geq 0, 0<\varpi\leq 1}; T_{\varpi}(t))$ on H for the model (4.5.5)-(4.5.6).*

Before starting the proof, it necessary to recall the following definition

Definition 4.5.3 (Favard space [11, Chap. III, Definition 5.10]). *1. Let $\{G(T_{\varpi})\}_{T_{\varpi} \geq 0}$ be a one-family strongly continuous semigroup and assume that $v_0 < 0$. For each $\nu \in (0, 1]$, the space*

$$F_{\nu} := \left\{ h \in H : \sup_{T_{\varpi} > 0} \left\| \frac{1}{(T_{\varpi})^{\nu}} (G(T_{\varpi})h - h) \right\| < \infty \right\}$$

equipped with norm

$$\|h\|_{F_{\nu}} := \sup_{T_{\varpi} > 0} \left\| \frac{1}{(T_{\varpi})^{\nu}} (G(T_{\varpi})h - h) \right\|$$

is called the Favard space of order ν.

2. Now for a general ν, let $\nu \in \mathbb{R}$, and choose $v > v_0$. We can write $\nu = q + \varphi$ where $q \in \mathbb{Z}$ and $0 < \varphi \leq 1$. Therefore the Favard space of order ν associated to the semigroup $\{G_A(T_{\varpi})\}_{T_{\varpi} \geq 0}$ is specified as the φ^{th} Favard space associated to the rescaled semigroup

$$\left\{ Exp \left[-v \left(\frac{\left(t + \frac{1}{\Gamma(\varpi)} \right)^{\varpi} - \Gamma(\varpi)^{-\varpi}}{\varpi} \right) \right] G_q(T_{\varpi}) \right\}_{T_{\varpi} \geq 0} = \left\{ e^{-vT_{\varpi}} G_q(T_{\varpi}) \right\}_{T_{\varpi} \geq 0}.$$

***Proof of Proposition** 4.5.2.* To show this result, we first consider the model (4.5.5)-

(4.5.6) where the delay vanishes (i.e. $\theta = 0$), leading to the associated operator $(\hat{A}, D(\hat{A}))$ given by

$$\hat{A}g := D_t^{\varpi} g, \tag{4.5.8}$$
$$D(\hat{A}) := \{g \in C([-\tau, 0], \mathcal{H}) : \; g(0) \in D(\mathbb{B}) \text{ and } D_t^{\varpi} g|_{t=0} = \mathbb{B}g(0)\}).$$

Referring to [16, Corollary 3.4], the operator $(\hat{A}, D(\hat{A}))$ is the generator of a strongly continuous two-parameter solution operator $(\{\hat{S}_{\varpi}(t)\}_{t\geq 0, 0<\varpi\leq 1}; T_{\varpi}(t))$ on $\mathcal{H}$. According to the Definition 4.3.2, we have $\hat{S}_{\varpi}(t) = \hat{G}_{\hat{A}}(T_{\varpi})$ for all times T_{ϖ}, $t > 0$ and hence, $\{\hat{G}_{\hat{A}}(T_{\varpi})\}_{T_{\varpi}\geq 0}$ is a one-family strongly continuous semigroup (in T_{ϖ}) generated by the operator $\hat{A}$, and explicitly reading as

$$[\hat{G}_{\hat{A}}(T_{\varpi})g](\kappa) = g(T_{\varpi} + \kappa), \quad \text{for} \quad T_{\varpi} + \kappa \leq 0$$

and

$$[\hat{G}_{\hat{A}}(T_{\varpi})g](\kappa) = G_{\mathbb{B}}(T_{\varpi} + \kappa)[g(0)], \quad \text{for} \quad T_{\varpi} + \kappa > 0$$

where $\{G_{\mathbb{B}}(T_{\varpi})\}_{T_{\varpi}\geq 0}$ is the one-family strongly continuous semigroup given in (4.5.4). Hence, the properties for one-family semigroup and strong continuity for $\{\hat{G}_{\hat{A}}(T_{\varpi})\}_{T_{\varpi}\geq 0}$ are directly induced by similar properties of $\{G_{\mathbb{B}}(T_{\varpi})\}_{T_{\varpi}\geq 0}$. We need now to check that the generator of $\{\hat{G}_{\hat{A}}(T_{\varpi})\}_{T_{\varpi}\geq 0}$ effectively coincides with $\hat{A}$. To achieve it we recall [16] that

$$T_{\varpi}(t) \longrightarrow 0 \text{ as } t \longrightarrow 0$$

and take for a sufficiently small $T_{\varpi} > 0$ and $\kappa \in [-\tau, 0]$ the difference quotient

$$\begin{aligned} &\left[\frac{\hat{G}_{\hat{A}}(T_{\varpi})g - g}{T_{\varpi}}\right](\kappa) = \frac{g(T_{\varpi} + \kappa) - g(\kappa)}{T_{\varpi}} \quad \text{for} \quad \kappa < 0 \\ &\text{and} \\ &\left[\frac{\hat{G}_{\hat{A}}(T_{\varpi})g - g}{T_{\varpi}}\right](\kappa) = \frac{G_{\mathbb{B}}(T_{\varpi})[g(0)] - [g(0)]}{T_{\varpi}} \quad \text{for} \quad \kappa = 0. \end{aligned} \tag{4.5.9}$$

Knowing that

$$\hat{A}g = \lim_{t\to 0} \frac{\hat{S}_{\varpi}(t)g - g}{t} = \lim_{T_{\varpi}\to 0} \frac{\hat{G}_{\hat{A}}(T_{\varpi})g - g}{T_{\varpi}},$$

it follows that limit of $\frac{\hat{G}_{\hat{A}}(T_{\varpi})g - g}{T_{\varpi}}$ exists in H as $t \searrow 0$ if and only if g is continuously differentiable with

$$g(0) \in D(\mathbb{B}) \quad \text{and} \quad D_t^{\varpi} g|_{t=0} = \mathbb{B}g(0),$$

and the checking is done.

Lastly, to conclude the proof we make use of a Desch-Schappacher perturbation [11] of $\hat{A}$ to generate A. For that we set

$$\Upsilon = (\hat{A} - \lambda) \cdot (I - \zeta_{\lambda} \odot R(\lambda, \mathbb{B})\theta)$$

where $\lambda \in \rho(\mathbb{B})$, the resolvent set of the operator $\mathbb{B}$, $R(\lambda, \mathbb{B})$ its resolvent at the point λ and $\zeta_\lambda \odot R(\lambda, \mathbb{B}) \in \mathcal{L}(\mathcal{H}, H)$ given by

$$([\zeta_\lambda \odot R(\lambda, \mathbb{B})]h)(\kappa) := \mathrm{e}^{\lambda\kappa} R(\lambda, \mathbb{B})h \quad \text{for} \quad h \in \mathcal{H}, \ \kappa \in [-\tau, 0].$$

Hence,

$$\begin{aligned} D(\Upsilon) &= \{g \in H : g - \zeta_\lambda \odot R(\lambda, \mathbb{B})\theta g \in D(\hat{A})\} \\ &= \{g \in C([-\tau, 0], \mathcal{H}) : \ g(0) \in D(\mathbb{B}) \text{ and } D_t^\varpi g|_{t=0} - \lambda R(\lambda, \mathbb{B})\theta g = \mathbb{B}\left(g(0) - R(\lambda, \mathbb{B})\theta g\right)\} \\ &= D(A), \end{aligned}$$

It follows that the following equality holds

$$A - \lambda = (\hat{A} - \lambda) \cdot (I - \zeta_\lambda \odot R(\lambda, \mathbb{B})\theta) \quad (= \Upsilon) \tag{4.5.10}$$

and also

$$\begin{aligned} \Upsilon g &= \left(\frac{d}{d\kappa} - \lambda\right) \cdot (I - \zeta_\lambda \odot R(\lambda, \mathbb{B})\theta)g \\ &= \left(\frac{d}{d\kappa} - \lambda\right) g \\ &= (A - \lambda)g, \end{aligned}$$

$\forall g \in D(\Upsilon) = D(A)$. Lets us now use the relation that exists between multiplicative and additive perturbations (see section: Additive Versus Multiplicative Perturbations [11, Chap. III, Proposition 3.18]) to obtain that

$$A = \hat{A}_{-1} - (\hat{A} - \lambda)_{-1}(\zeta_\lambda \odot R(\lambda, \mathbb{B})\theta)|_H$$

where $\hat{A}_{-1}$ is the extrapolated operator taken from corresponding space of negative order (see [11, Chap. II, Section 5.a]). To complete the proof, we can use [11, Chap. III, Corrollary 3.6] to demonstration that the perturbing operator maps into the extrapolated Favard space (Definition 4.5.3) associated to $\hat{A}_{-1}$, that is the range

$$\Gamma := \ \mathrm{rg}\,[(\zeta_\lambda \odot R(\lambda, \mathbb{B}))\theta] \subset F_1^{\hat{A}}.$$

To achieve it, let us take any $g = \zeta_\lambda \odot h \in \Gamma$, and prove that it is also in $F_1^{\hat{A}}$. Hence, $h \in D(\mathbb{B})$ and the quotient (4.5.9) yields

$$\overline{\lim_{T_\varpi \searrow 0}} \left\| \frac{\hat{G}_{\hat{A}}(T_\varpi)g - g}{T_\varpi} \right\| \leq \max\{\|\lambda\zeta_\lambda\|_\infty \cdot \|h\|, \ \|\mathbb{B}h\|\} < \infty,$$

meaning that $g \in F_1^{\hat{A}}$ ■

4.6 Numerical approximations and simulations

We can now perform some numerical simulations in order to examine the evolution dynamics for population models of type (4.5.1) in the context of aforementioned analysis. The model reads as

$$D_t^{\varpi} u(t) = -bu(t) + au(t-\tau) \tag{4.6.1}$$

and is assumed to satisfy the initial conditions

$$u(0) = f. \tag{4.6.2}$$

We utilize the implementation code of the predictor-corrector PECE technique of Adams-Bashforth-Moulton type described in [17] to provide numerical approximations of solutions to the population system (4.6.1)-(4.6.2) with different values of ϖ. The evolution dynamics in the time interval [0, 25] for solutions to (4.6.1)-(4.6.2) are shown in Fig. 2 (a) and Fig. 2 (b) for $\varpi = 1.00$ and $\varpi = 0.90$ respectively. Different delays are considered in both cases (integer and pure fractional order), but the figures exhibit population evolution with similar trajectories. This concludes and validates the bounded perturbation theory mentioned and detailed above.

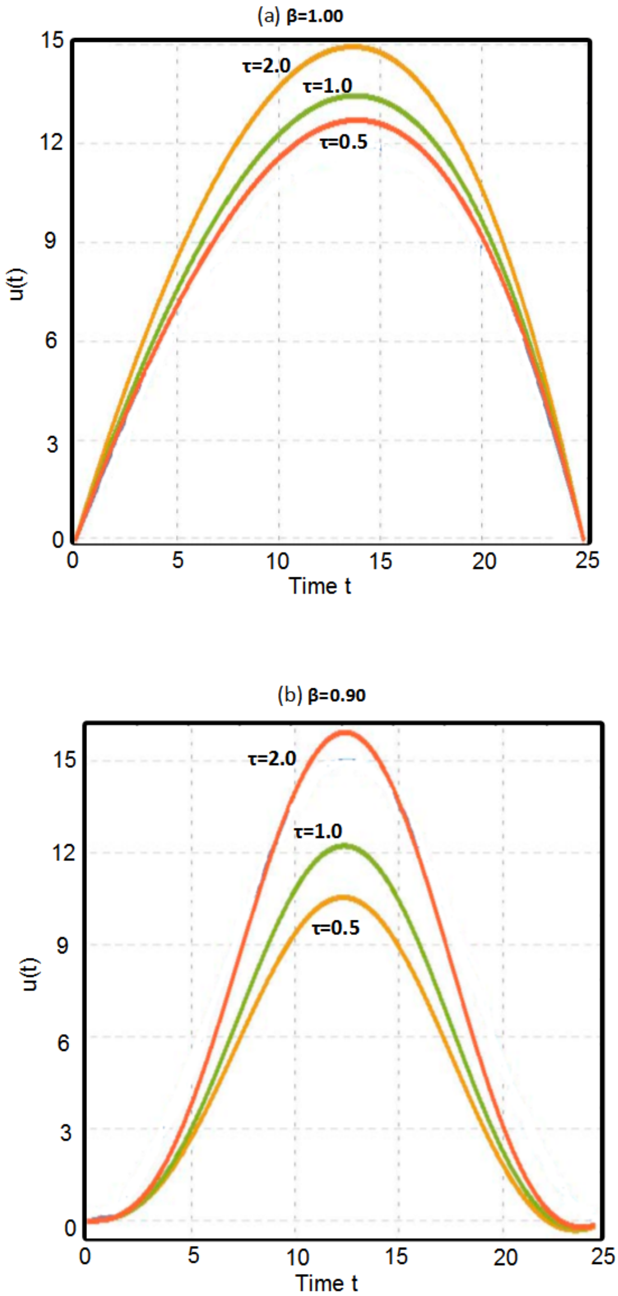

Figure 4.2: Numerical solutions illustrating the development in the time interval [0, 25] for solutions to (4.6.1)-(4.6.2). Various delays are considered and analogous trajectories are appeared in the two figures (integer and fractional order).

Chapter 5

Conclusion and open problem

The entire research, investigation and analysis in this dissertation have been about applying linear evolution equations to different types of the derivatives, especially ϖ-derivative which is the main objective of this dissertation. In chapter 3 we applied linear evolution equations to Newtonian and old Caputo fractional derivatives, It was found that some problems can not be solved utilizing the above derivatives. Hence, making use of ϖ-derivative, a locally defined derivative with a fractional order ϖ, assumed to be less than one ($0 < \varpi < 1$). We have proved the classical bounded perturbation theorem for two-parameter solution operators of evolution equations with a fractional parameter and the application to a model of delay population dynamics with numerical simulations has been provided as a validation of the theory. This is the first instance where such assertion is proven for evolution equations with a fractional order derivative. In fact, the assertion is not in general true for evolution models using the most popular existing versions of fractional derivatives, like Caputo's derivative or Riemann-Liouville's derivative. Furthermore, the analysis presented here is the extension of evolution equations using the conventional first order derivative.

Finally, due to the fact that some perturbations, like for instance, perturbation by unbounded operator, Miyadera perturbation, Kato's perturbation and perturbation of dissipative operators remain open problems in the field of fractional calculus [10, 14, 18], the results obtained in this research will certainly lead to more investigations and pave the way for more advanced studies, like perturbation and the stability of the results for models with ϖ-derivative.

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Appendix A

Appendix

A.1 Non-autonomous fragmentation model

Here we show how to apply Non-autonomous fragmentation model using homogeneous initial valve problem. This type of model is fully discussed in [19, 30].

Consider the following non-autonomous evolution of the mass distribution $u(t,s)$ described, by the linear integro-differential system:

$$\begin{cases} \frac{\partial}{\partial t}u(t,s) = -a(t,s)u(t,s) + \int_s^\infty a(t,y)b(s|y)u(t,y)dy \\ u(\tau,s) = u_\tau(s) \qquad 0 \le \tau < t \le V, \quad s > 0; \end{cases} \tag{A.1.1}$$

see[30] where u is the particle mass distribution function $u(\tau,s) = u_\tau(s)$ is the mass distribution at some fixed time $\tau \ge 0)$ with respect to the mass s, $b(s|y)$ is the distribution of particle masses s spawned by the fragmentation of a particle of mass y. at the time $t < V \in \mathbb{R}$ and $a(t,s)$ is the evolutionary time dependent fragmentation rate, that is the rate at which mass s particles break up at a time t. The first term on the right-hand side of (A.1.1) describes the reduction in the number of particles in the mass range $(s : s + ds)$ due to the fragmentation of particles in the same range. The second term describes the increase in the number of particles in the range due to fragmentation of larger particles. The idea here is to analyze the equation (A.1.1) in the Banach space $L_1(\mathbb{J}, Z_1)$ where $\mathbb{J} = [0,V]$ and

$$Z_1 = L_1([0,\infty), sds) = \{\psi : ||\psi||_{Z_1} := \int_0^\infty s|\psi(s)|ds < \infty\}$$

using the theory of evolution semi-group.
Throughout, we will consider the following regularity assumptions:
$(t,s) \to a(t,s) \in L_1([0,V'], L_\infty([k,l])) \; \forall 0 < k < l < \infty$ and $V' \in (0,V)$,
$b(s|y)$ is a positive measurable function with

$$b(s|y) = 0 \quad \forall s \ge y \quad and \quad 0 \le t \le V,$$

and the local conservative law

$$\int_0^y sb(s|y)ds = y$$

for all $y \geq 0$ and $0 \leq t \leq V$.
The model (A.1.1) is recast as the non-autonomous abstract Cauchy problem in Z_1

$$\begin{cases} \frac{du(t)}{dt} = P(t)u(t) \qquad for \quad 0 \leq \tau < t \leq V \\ u(\tau) = u_\tau \end{cases} \tag{A.1.2}$$

where $P(t)$ is defined by $P(t) = \mathcal{P}(t)$ and represent the realization of $\mathcal{P}(t)$ on the domain

$$D(P(t)) = \{u \in Z_1 : \mathcal{P}(t)u(t) \in Z_1\},$$

with $(\mathcal{P}u)$ defined as

$$(\mathcal{P}u)(t,s) = (\mathcal{P}u)(t)(s) = -a(t,s)u(t,s) + \int_s^\infty a(t,y)b(s|y)u(t,y)dy$$

$\mathcal{P}(t)$ is seen as the pointwise operation

$$\psi(t,s) \to -a(t,s)\psi(t,s) + \int_s^\infty a(t,y)b(s|y)\psi(t,y)dy$$

defined on the set of measurable function. $\mathcal{P}(t)$ indeed defines various operators.

Going back to abstract Cauchy problem (A.1.2), Using (Theorem (3.4.1)), it is clear that for $0 \leq t \leq V$. $P(t)$ is a bounded linear operator in Z_1 and that $t \to P(t)$ is continuous in the uniform operator topology. Next we will find the propagator $U(t,\tau)$, such that $u(t) = U(t,\tau)u_\tau$ is in the sense, a solution of (A.1.2) satisfying the initial condition $u(\tau) = u_\tau$. Hence we have the following

Lemma A.1.1 ([30]). *Let $P(t)$ be a bounded linear operator in Z_1 for $0 \leq t \leq V$. If the function $t \to P(t)$ is continuous in the uniform operator topology, then for every $u_\tau \in Z_1$, the abstract Cauchy problem (A.1.2) has a unique classical solution u given by the relation:*

$$u(t) = u_\tau + \int_\tau^t P(\zeta)u(\zeta)d(\zeta). \tag{A.1.3}$$

Proof. [10, Theorem 5.1, Chapter 5], ■

the proof is done in a Banach space Z which is also true in Z_1

Theorem A.1.2 ([30]). *There is a propagator $U(t,\tau)$ associated with the initial value problem (A.1.2) such that $U(t,\tau)u_\tau$ is its solution satisfying the initial condition $u(\tau) = u_\tau$.*

Proof. From Lemma (A.1.1), the existence and uniqueness of the solution can already be noticed. Let $u(t)$ be this solution. The so- called solution operator of (A.1.2) is defined by

$$U(t,\tau)u_\tau = u(t) \qquad for \quad 0 \le \tau < t \le V \tag{A.1.4}$$

- For every $u_\tau \in Z_1$, $U(\tau,\tau)u_\tau = u(\tau) = u_\tau$, then $U(\tau,\tau) = I$ (condition(i)).
- For every $u_\tau \in Z_1$, we have $U(t,\tau)u_\tau = u(t)$ and $U(t,r)U(r,\tau)u_\tau = U(t,r)u(r) = u(t)$, then condition (ii) follows from the uniqueness of the solution of (A.1.2).
- It is obvious that $U(t,\tau)$ is a linear operator defined in all Z_1 since (A.1.2) is linear . The relation (A.1.3) implies $||u(t)|| \le ||u_\tau|| + \int_\tau^t ||P(\zeta)||\ ||u(\zeta)||\ d\zeta$ and from Gronwall's inequality we also have $||u(t)|| \le ||u_\tau|| exp(\int_\tau^t ||P(\zeta)||\ d\zeta)$. Then (A.1.4) yields $||U(t,\tau)u_\tau|| = ||u_\tau|| exp(\int_\tau^t ||P(\zeta)||\ d\zeta)$, leading to

$$||U(t,\tau)|| = exp(\int_\tau^t ||P(\zeta)|| d\zeta).$$

Where $U(t,\tau)$ is bounded and, therefore, strongly continuous. This concludes the proof. ■

The fact that $P(t)$ is bounded makes this existence results easier to obtain. Unfortunately, $P(t)$ is not all ways bounded, for Equivalent norm approach and more details the reader can consult [19, 30].

A.2 Evolution for transport-convection dynamics with a New Parameter: An alternative method.

Here we use alternative techniques (the two-parameter matrix solution operators) address the well-posedness of the transport-convection models with a new parameter of the type

$$ {}_0^A D_t^\beta p(t,x,n) = -div(\omega(x,n)p(t,x,n)) - a_n p(t,x,n) + \sum_{m=n+1}^{\infty} b_{n,m} a_m p(t,x,m), \tag{A.2.1}$$

where $t > 0,\ 0 < \beta \le 1,\ x \in \mathbb{R}^3,\ \ n = 1,2,3,...$ and subject to initial conditions

$$p(0,x,n) = \overset{o}{p}_n(x), \qquad n = 1,2,3,... \tag{A.2.2}$$

The concepts defined in Chapter 2 are used, especially the derivative with a new parameter ${}_0^A D_t^\beta$. The model (A.7.1) may take the generalised form

$$\begin{aligned} {}_0^A D_t^\beta u(x,t) &= [\mathbb{A}u(\cdot,t)](x), \quad 0 < \beta \le 1,\ \ x,\ t > 0 \\ u(x,0) &= \tilde{f}(x)\ ,\ \ x > 0, \end{aligned} \tag{A.2.3}$$

where $\mathbb{A}$ is a certain differential and (or) integral expression, that can be evaluated at any point $x > 0$ for functions u belonging to a certain subset of the domain of $\mathbb{A}$.

A.3 Two-parameter matrix solution operators

To proceed we can define a Banach space H endowed with the norm $\|\cdot\|_H$, express the model (A.2.3) in the form

$$\begin{aligned} &{}_0^A D_t^\beta u(t) = Au(t), \quad 0 < \beta \leq 1, \;\; t > 0 \\ &u(0) = f \end{aligned} \tag{A.3.1}$$

and define the domain

$$D(A) := \{v \in H : Av \in H\} \tag{A.3.2}$$

on which the realization operator A of the expression $\mathbb{A}$ is defined. To study (A.3.1), we can exploit the differential system

$$\begin{aligned} &{}_0^A D_t^\beta u(t) = \mu u(t), \quad 0 < \beta \leq 1, \;\; t > 0, \;\; \mu \in \mathbb{C} \\ &u(0) = f_0. \end{aligned} \tag{A.3.3}$$

It is easy to check that, instead of Mittag-Leffler function or one of its variants, the following expression new in the literature, uniquely solves the model (A.3.3):

$$\mathcal{E}_\beta(t) = f_0 \exp\left[\mu\left(\frac{\left(t + \frac{1}{\Gamma(\beta)}\right)^\beta - \Gamma(\beta)^{-\beta}}{\beta}\right)\right]. \tag{A.3.4}$$

We note that for $\beta = 1$ the following well known classical result holds:

$$u(t) = f_0 e^{\mu t}.$$

Remark A.3.1. *If we set a certain* $T_\beta = T = \frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta - \Gamma(\beta)^{-\beta}}{\beta}$*, then the expression* $v(T) = f_0 e^{\mu T}$ *uniquely solves*

$$\begin{aligned} &\partial_T u(T) = \mu u(T), \;\; t > 0, \;\; \mu \in \mathbb{C} \\ &u(0) = f_0, \end{aligned} \tag{A.3.5}$$

where ∂_T *means partial derivative (normal derivative) with respect to* T*. Hence, the expression (A.3.4) uniquely solves (A.3.3) always implies that there exists a function at least in* $C(\mathcal{R}_+, H) \cap C^1(\mathcal{R}_+, H)$ *solving (A.3.5)*

This remark will be very important in our analysis, with a special attention to the expression of T. Next we consider the system of linear differential equations using the β–derivative with constant coefficients:

$$\begin{aligned} &{}_0^A D_t^\beta u_1 = \mu_{11} u_1 + \mu_{12} u_2 + \cdots + \mu_{1n} u_n, \\ &\vdots \\ &{}_0^A D_t^\beta u_n = \mu_{n1} u_1 + \mu_{n2} u_2 + \cdots + \mu_{nn} u_n, \end{aligned} \tag{A.3.6}$$

where $0 < \beta \leq 1,\ \ t > 0,\ \mu \in \mathbb{C}$. The linearity of the operator ${}_0^A D_t^\beta$ allows us to write the system (A.3.6) in the matrix form

$$ {}_0^A D_t^\beta U(t) = MU(t) \tag{A.3.7}$$

with U is a n–vector whose components are the unknown functions u_i and M is the $n \times n$ matrix $(\mu_{ij})_{1 \leq i,j \leq n}$. Let $U(0) = U_0$ be the initial condition vector for (A.3.7). We extend Peano's idea [?] by stating by analogy to solution (A.3.4) that the system (A.3.7) can be solved using explicitly the formula

$$U(t) = \exp\left[\left(\frac{\left(t + \frac{1}{\Gamma(\beta)}\right)^\beta - \Gamma(\beta)^{-\beta}}{\beta}\right) M\right] U_0 \tag{A.3.8}$$

where the matrix exponential

$$\exp\left[\left(\frac{\left(t + \frac{1}{\Gamma(\beta)}\right)^\beta - \Gamma(\beta)^{-\beta}}{\beta}\right) M\right] = \exp\left[T_\beta M\right] = I + \frac{T_\beta M}{1!} + \frac{T_\beta^2 M^2}{2!} + \cdots \tag{A.3.9}$$

with

$$T_\beta = T_\beta(t) = \frac{\left(t + \frac{1}{\Gamma(\beta)}\right)^\beta - \Gamma(\beta)^{-\beta}}{\beta} \tag{A.3.10}$$

Remark A.3.2. *It is easy to see that the function*

$$\begin{aligned} T_\beta :\quad & \mathbb{R}_+ \longrightarrow \mathbb{R}_+. \\ & t \longmapsto \frac{\left(t + \frac{1}{\Gamma(\beta)}\right)^\beta - \Gamma(\beta)^{-\beta}}{\beta}, \quad 0 < \beta \leq 1 \end{aligned}$$

is a topological homeomorphism from $\mathbb{R}_+$ *to* $\mathbb{R}_+$. *Thus, the topological properties of the space* $\mathbb{R}$ *(endowed with a topology) is preserved when transforming* t *to* $T_\beta(t)$

Now, we consider the space $\mathfrak{M}_n(\mathbb{C})$ of all complex $n \times n$ matrices and endowed with the matrix-norm. By definition we have

$$\exp\left[\left(\frac{\left(t + \frac{1}{\Gamma(\beta)}\right)^\beta - \Gamma(\beta)^{-\beta}}{\beta}\right) M\right] = \exp\left[T_\beta M\right] = \sum_{k=0}^{\infty} \frac{T_\beta^k M^k}{k!} \tag{A.3.11}$$

for all $M \in \mathfrak{M}_n(\mathbb{C})$ and $0 < \beta \leq 1$. It is well known and not difficult to show that the partial sums of the series (A.3.11) form a Cauchy sequence, and so, the series converges.

Proposition A.3.3. *For any $M \in \mathfrak{M}_n(\mathbb{C})$ and $0 < \beta \leq 1$, the map*

$$\begin{array}{ccc} \mathbb{R}_+ & \longrightarrow & \mathfrak{M}_n(\mathbb{C}) \\ t & \longmapsto & \exp\left[\left(\dfrac{\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta} - \Gamma(\beta)^{-\beta}}{\beta}\right) M\right] \end{array} \tag{A.3.12}$$

is continuous.

Proof. The proof follows from the fact that the map $T_\beta \longmapsto \exp[T_\beta M]$ is continuous in T_β and completed by Remark A.3.2. ■

The following well known results [11] that apply for exponential functions holds

Proposition A.3.4. *For any $M \in \mathfrak{M}_n(\mathbb{C})$ and $0 < \beta \leq 1$,*

$\exp[(T_\beta + S_\beta)M] = \exp[T_\beta M] \cdot \exp[S_\beta M]$

$\exp[0M] = I.$

Hence, the map $T_\beta \longmapsto \exp[T_\beta M]$ is a homomorphism of the additive semigroup $(\mathbb{R}_+, +)$ into a multiplicative semigroup of matrices $(\mathfrak{M}_n, \cdot)$.

Definition A.3.5. *The modified time expressed by T_β in (A.3.10) is called the revamped time (or GA-revamped time) corresponding to t for the model (A.3.7)*

Remark A.3.6. *Note that $T_\beta : \mathbb{R}_+ \to \mathbb{R}_+$ is well defined and increasing for $0 < \beta \leq 1$ with*

- $T_\beta(0) = 0$
- $T_1(t) = t$
- $\frac{dT_\beta(t)}{dt} = \left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta - 1}$

This means the revamped time always coincide with its corresponding time at the beginning (initial conditions) or when $\beta = 1$ (coventional first order derivative).

Definition A.3.7 (**Two-parameter matrix solution operators**). *Let us fix $\beta \in (0, 1]$ and $t \in \mathbb{R}_+$. The pair $(\{S_\beta(t)\}_{t \geq 0, 0<\beta\leq 1}; T_\beta(t))$ where $T_\beta(t) = \frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta} - \Gamma(\beta)^{-\beta}}{\beta}$, is called the two-parameter matrix solution operator for the system (A.3.7), where $\{S_\beta(t)\}_{t\geq 0, 0<\beta\leq 1}$ is the two-parameter family such that*

- *$S_\beta(t) = G(T_\beta)$ with T_β the revamped time corresponding to t.*
- *$\{G(T_\beta)\}_{T_\beta \geq 0}$, the one-parameter family defined as*

$$G(T_\beta) = \exp\left[\left(\frac{\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta} - \Gamma(\beta)^{-\beta}}{\beta}\right) M\right] = \exp[T_\beta M] \tag{A.3.13}$$

and representing a semigroup (in T_β) generated by the matrix $M \in \mathfrak{M}_n(\mathbb{C})$,

A.4 Strongly continuous two-parameter solution operators

With the previous definition in mind, we come back to the model (A.3.1):

$$\begin{aligned} {}_0^A D_t^\beta u(t) &= Au(t), \quad 0 < \beta \leq 1, \;\; t > 0. \\ u(0) &= f \end{aligned} \tag{A.4.1}$$

If $A : H \to H$ is a bounded linear operator then, we can exploit the Definition A.3.7 to solve the model (A.4.1) together with the exponential series represented in (A.3.11), which is still convergent with respect to the norm in the space of bounded linear operators $\mathcal{B}(H)$. In this case, the pair $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}, T_\beta(t))$ defined in the Definition A.3.7 and that solves (A.4.1) is simply called the two-parameter solution operator for the system (A.4.1). More precisely we have

Theorem A.4.1. *For the system (A.4.1), every uniformly continuous two-parameter solution operator*
$(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$ on a Banach space H induces a solution that is in the form (A.3.13):

$$u(t) = G(T_\beta)f = \exp\left[\left(\frac{\left(t + \frac{1}{\Gamma(\beta)}\right)^\beta - \Gamma(\beta)^{-\beta}}{\beta}\right) A\right] f, \qquad f \in D(A),$$

for some bounded linear operator A.

Proof. The proof follows from the previous section and the only point to add is that if $A : H \to H$ be a bounded linear operator, then the series

$$\sum_{k=0}^{\infty} \frac{\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta - \Gamma(\beta)^{-\beta}}{\beta}\right)^k A^k}{k!}$$

converges in the used norm for every $t > 0$. ∎

However, the reality is sometime complex and as mentioned in the introduction, the operator A is, in most of the cases, unbounded. Simple examples are differential operators that are not bounded on the whole space H. Then multiple iterates of operator A appearing in the series (A.3.11) make it impossible to use the series to solve (A.4.1). The main reason is that the common domain of those iterates of A could reduced to the null subspace $\{0\}$. Then, more considerations, in addition to what was developed in the previous section are necessary.

Definition A.4.2 (Strongly continuous two-parameter solution operators). *Let us fix $\beta \in (0, 1]$ and $t \in \mathbb{R}_+$. The pair $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$ is said to be a strongly continuous two-parameter solution operator for the system (A.4.1) if the two-parameter family $\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}$ is such that*

- $S_\beta(t) = G(T_\beta)$ *with* T_β *the revamped time corresponding to* t.
- $\{G_A(T_\beta)\}_{T_\beta \geq 0}$ *is a strongly continuous semigroup (in* T_β*) generated by the operator* A*, that is*

(i) $G_A(0) = I$;

(ii) $G_A(T_\beta + S_\beta) = G_A(T_\beta)G_A(S_\beta)$ *for all* $T_\beta, S_\beta \geq 0$;

(iii) $\lim_{T_\beta \to 0^+} G_A(T_\beta)f = f$ *for any* $f \in H$.

Remark A.4.3. *Note that:*

(a) *For* $\beta = 1$, $T_\beta(t) = t$ *and the definition here above coincides with the definition of the classical well known (one-parameter)* C_0*-semigroup.*

(b) *If* $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$ *is a strongly continuous two-parameter solution operator for the system (A.4.1) generated by* A*, then,*

$$Af = \lim_{t\to 0} \frac{S_\beta(t)f - f}{t} = \lim_{T_\beta \to 0} \frac{G_A(T_\beta)f - f}{T_\beta}, \tag{A.4.2}$$

where the domain of A*,* $D(A)$*, is chosen to be defined as the set of all* $f \in H$ *for which this limit exists.*

The later equality is due to the above Definition A.4.2 and the fact that $T_\beta(t) \to 0$ *as* $t \to 0$.

(c) *–If* $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$ *is a strongly continuous two-parameter solution operator for the system (A.4.1) generated by* A*, then, for* $f \in D(A)$ *the function* $t \longrightarrow S_\beta(t)f = G_A(T_\beta)f$ *is a classical solution of the fractional Cauchy problem (A.4.1).*

–For $f \in H \setminus D(A)$*, however, the function* $u(t) = S_\beta(t)f$ *is continuous but, in general, not differentiable, nor* $D(A)$*-valued, and, therefore, not a classical solution.*

(d) *The strongly continuous two-parameter solution operator* $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$ *is bounded in the operator norm over any compact interval of* $\mathbb{R}_+$ *thanks to properties (ii) and (iii) here above and the Banach–Steinhaus theorem which show that any* C_0*-semigroup like* $\{G_A(T_\beta)\}_{T_\beta\geq 0}$ *is bounded in the operator norm over any compact interval of* $\mathbb{R}_+$.

(e) *If* $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$ *is a strongly continuous two-parameter solution operator for the system (A.4.1) generated by* A*, then, for* $f \in D(A)$ *the function* $T_\beta \longrightarrow G_A(T_\beta)f$ *a classical solution of*

$$\begin{aligned} \partial_t u(t) &= Au(t), \quad t > 0. \\ u(0) &= f \end{aligned} \tag{A.4.3}$$

More precisely, we have the following statement:

Proposition A.4.4. *Let* $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$ *be the a strongly continuous two-parameter solution operator for the system (A.4.1) generated by* $(A, D(A))$*. Then* $t \to S_\beta(t)f = G_A(T_\beta)f, \quad f \in D(A)$*, is the only solution of (A.4.1) taking values in* $D(A)$*.*

Proof. To prove it we set $u(t) = v(T_\beta) \in D(A)$ for all $t > 0$, where $T_\beta = T_\beta(t)$ is the revamped time corresponding to t, $v \in C(\mathcal{R}_+, H) \cap C^1(\mathcal{R}_+, H)$ and ${}_0^A D_t^\beta u(t) = Au(t), \quad t > 0$. Then, by the Definition (A.4.2), $v(T_\beta)$ satisfies $\partial_t u(t) = Au(t), \quad t > 0$. Let us define the function

$$\begin{array}{rccc} z: & (0, T_\beta) & \longrightarrow & H \\ & S_\beta & \longmapsto & G_A(T_\beta - S_\beta) v(S_\beta) \end{array}$$

and make use of the well known property of semigroups [11]:

$$\partial_{T_\beta} G_A(T_\beta) v(T_\beta) = AG_A(T_\beta) v(T_\beta) = G_A(T_\beta) Av(T_\beta),$$

to state that z is differentiable and

$$0 = \partial_{S_\beta} z(S_\beta) = G_A(T_\beta - S_\beta)(\partial_{S_\beta} v(S_\beta) - (Av)(S_\beta)). \tag{A.4.4}$$

Thus, z is constant on $(0, T_\beta)$, meaning that for any $\varepsilon, \eta \in (0, T_\beta)$ we have

$$G_A(T_\beta - \varepsilon) v(\varepsilon) = G_A(T_\beta - \eta) v(\eta)$$

which tends to

$$G_A(T_\beta) v(0) = v(T_\beta)$$

as ε tends to 0 and η tends to T_β. This proves that v is defined by the semigroup $\{G_A(T_\beta)\}_{T_\beta \geq 0} = \{S_\beta(t)\}_{t \geq 0, 0 < \beta \leq 1}$. Hence, by the Definition (A.4.2), u is also defined by the strongly continuous two-parameter solution operator $\{S_\beta(t)\}_{t \geq 0, 0 < \beta \leq 1}$, which concludes the proof. ■

It is now clear that for $f \in D(A)$,

$$D_t^\beta S_\beta(t) f = \frac{d}{dT_\beta} G_A(T_\beta) f.$$

Hence, making use of the well know properties of strongly continuous semigroups, we have the following corollary

Corollary A.4.5. *Let $(\{S_\beta(t)\}_{t \geq 0, 0 < \beta \leq 1}; T_\beta(t))$ be the a strongly continuous two-parameter solution operator for the system (2.7.2) generated by $(A, D(A))$. Then, for $f \in D(A)$, $\quad S_\beta(t) f \in D(A)$ and*

$$D_t^\beta S_\beta(t) f = AS_\beta(t) f = S_\beta(t) Af. \tag{A.4.5}$$

for all $t \geq 0$.

Definition A.4.6. *(**Two-parameter solution operators β-exponentially bounded**)*

- *The strongly continuous two-parameter solution operator $(\{S_\beta(t)\}_{t \geq 0, 0 < \beta \leq 1}; T_\beta(t))$ for the system (A.4.1) is said to be β-exponentially bounded if there exist constants $\omega \geq 0$ and $M \geq 1$ such that*

$$\|S_\beta(t)\|_H \leq M \exp\left[\omega \left(\frac{\left(t + \frac{1}{\Gamma(\beta)}\right)^\beta - \Gamma(\beta)^{-\beta}}{\beta}\right)\right] \tag{A.4.6}$$

- *If the system (A.4.1) admits a strongly continuous two-parameter solution operator* $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$ *satisfying (A.4.6), then we say that the operator* $A \in \mathcal{G}^\beta(M,\omega)$.
- $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$ *is said to be contractive if*

$$\|S_\beta(t)\|_H \leq 1, \tag{A.4.7}$$

and we say $A \in \mathcal{G}^\beta(1,0)$.

- *As in [18], we say that the problem (A.4.1) is well-posed if it admits a strongly continuous two-parameter solution operator.*

Let us set

$$\mathcal{G}^\beta(\omega) := \bigcup\{\mathcal{G}^\beta(M,\omega), M \geq 1\},$$

$$\mathcal{G}^\beta := \bigcup\{\mathcal{G}^\beta(\omega), \omega \geq 0\}$$

and denote by

$$\mathcal{B}(H) := \mathcal{B}(H;H)$$

the space of all bounded linear operators from H to H.

Remark A.4.7. *The condition (A.4.6) holds if and only if the one parameter family* $\{G_A(T_\beta)\}_{T_\beta\geq 0}$ *given in the Definition (A.4.2) satisfies*

$$\|G_A(T_\beta)\|_H \leq Me^{\omega T_\beta} \tag{A.4.8}$$

Corollary A.4.8. *The problem (A.4.1) is well-posed if* $A \in \mathcal{B}(H)$

Proof. This is a direct consequence of Theorem A.4.1 and Proposition A.4.4. ∎

Next let us recall the following definition:

Definition A.4.9. *The set* $\rho(A)$ *is called the resolvent set of the operator* A *and is defined as*

$$\rho(A) = \{\lambda \in \mathbb{R}; \quad \lambda I - A : D(A) \to X \text{ is invertible and } \quad (\lambda I - A)^{-1} \in \mathcal{B}(H)\}. \tag{A.4.9}$$

Then, For $\lambda \in \rho(A)$, *the inverse* $R(\lambda, A) := (\lambda I - A)^{-1}$ *is, by the closed graph theorem, a bounded operator on* H *and is termed as the resolvent of* A *at the point* λ.

Proposition A.4.10. *If the strongly continuous two-parameter solution operator* $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$ *for the system (A.4.1) is* β*-exponentially bounded in terms of Definition A.4.6 then,* $S_\beta(t)$ *is related to its resolvent by the formula*

$$R(\lambda, A)f = \int_0^\infty \left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta-1} \exp\left[-\lambda\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta - \Gamma(\beta)^{-\beta}}{\beta}\right)\right] S_\beta(t) f dt, \tag{A.4.10}$$

for $f \in H$ *and* $Re\lambda > \omega$.

Proof. The proof follows from the Definition A.4.2 where $\{G_A(T_\beta)\}_{T_\beta \geq 0}$ is a strongly continuous semigroup with the operator A as infinitesimal generator and satisfying (A.4.8). Then, from the semigroup theory we have that

$$R(\lambda, A) = \int_0^\infty e^{-\lambda T_\beta} G_A(T_\beta) dT_\beta.$$

Substituting the revamped time T_β and using the Remark A.3.6 lead to the formula. ■

We can therefore propose the following diagram for the system (A.4.1) presenting the relations between the two-parameter solution operator $(\{S_\beta(t)\}_{t\geq 0, 0<\beta\leq 1}; T_\beta(t))$ its generator and its resolvent.

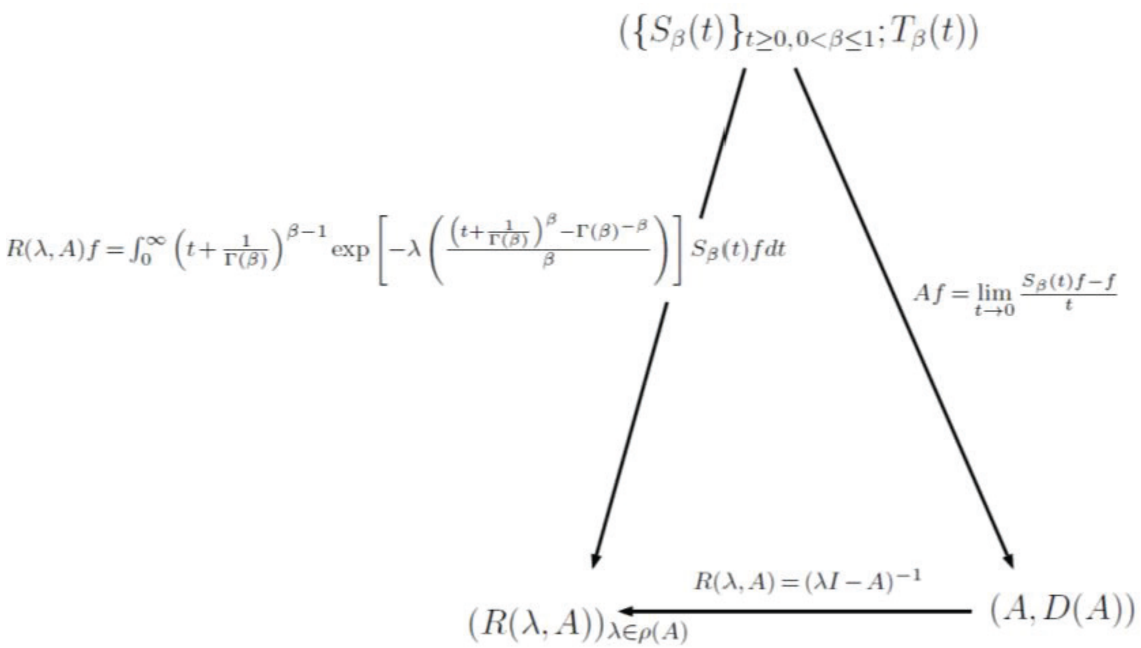

Figure A.1: Relations between the two-parameter solution operator, its generator and its resolvent

A.5 Exponential approximation and application

For dynamical systems (A.4.1) with unbounded operators A, analysis can be done by using the exponential approximation

$$\exp\left[\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta - \Gamma(\beta)^{-\beta}}{\beta}\right) A\right] f = \lim_{p\to\infty}\left[I - \frac{1}{p}\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta - \Gamma(\beta)^{-\beta}}{\beta}\right) A\right]^{-p} f. \tag{A.5.1}$$

If the above limit exists then, it defines a strongly continuous two-parameter solution operator as given in Definition (A.4.2). Conditions of the existence of the limit (A.5.1) are given by making use of the Hille–Yosida theorem (see [11, Chap II, Section 3]) in the theory of semigroup and completed by the Remark A.4.7. Then we have the following theorem that applies to the model (A.4.1) with the fractional parameter β;

Theorem A.5.1. *$A \in \mathcal{G}^{\beta}(M,\omega)$ if and only if (a) A is closed and densely defined, (b) there exist $M > 0, \omega \in \mathbb{R}$ such that $(\omega, \infty) \in \rho(A)$ and for all $n \geq 1$, $\lambda > \omega$,*

$$\|(\lambda I - A)^{-n}\| \leq \frac{M}{(\lambda - \omega)^n}. \tag{A.5.2}$$

where $\rho(A)$ is the resolvent set of the operator A as defined above.

Proposition A.5.2. *Let $(\{S_\beta(t)\}_{t\geq 0, 0<\beta\leq 1}; T_\beta(t))$ be the a strongly continuous two-parameter solution operator for the system (A.4.1) generated by A. Then*

$$S_\beta(t)f = \lim_{p\to\infty}\left[I - \frac{1}{p}\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta} - \Gamma(\beta)^{-\beta}}{\beta}\right)A\right]^{-p} f, \quad \textit{for } f \in H,$$

and the limit is uniform in t on any bounded interval.

Proof. Considering the revamped time corresponding to t, $T_\beta = T_\beta(t)$, we have by definition $S_\beta(t)f = G_A(T_\beta)f$. Since the one parameter family $\{G_A(T_\beta)\}_{T_\beta\geq 0}$ is a C_0-semigroup generated by A, we make use of [11, Corollary III 5.5] to write

$$G_A(T_\beta)f = \lim_{p\to\infty}\left(I - \frac{T_\beta}{p}A\right)^{-p} f, \quad \text{for } f \in H$$

and the proposition is proved. ∎

As application, we can approximate the solution for the system (A.4.1), by considering the alternate model given by

$$\begin{aligned}
&\frac{u_p\left[k\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}-\Gamma(\beta)^{-\beta}}{p\beta}\right)\right] - u_p\left[(k-1)\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}-\Gamma(\beta)^{-\beta}}{p\beta}\right)\right]}{\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}-\Gamma(\beta)^{-\beta}}{p\beta}\right)} = Au_p\left[k\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}-\Gamma(\beta)^{-\beta}}{p\beta}\right)\right]\\
&u_p(0) = f
\end{aligned} \tag{A.5.3}$$

for $0 < \beta \leq 1$, $t > 0$. The explicit solution of the problem (A.5.3) is given by

$$u_p(t) = \left[I - \left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta} - \Gamma(\beta)^{-\beta}}{p\beta}\right)A\right]^{-p} f$$

which represents an approximation of the solution for the model (A.4.1). Making use of Proposition A.5.2, we see that $\lim_{p\to\infty} u_p(t) = S_\beta(t)f$. Hence, the difference system (A.5.3) is very important in solving the model (A.4.1) since their solutions converge to the solution of (A.4.1) and from Proposition A.4.4, this solution $S_\beta(t)f$ is unique if f is taken from $D(A)$.

A.6 Subordination & prolongation principles with $\beta-$ derivative

In this section, we address the issue of subordination principle for evolution equations with fractional parameters. This principle has been proved only for models with Caputo fractional derivative [12, 18] and the opposite principle has been proved not to be true. Hence, we go farther by also addressing the opposite principle, named here the prolongation principle. Recall that these principles study existence of two-parameter solution operators for problems (2.7.2) with different values of derivative orders. We note that if we have a strongly continuous semigroup $\{G_A(T)\}_{T\geq 0}$ generated by the operator A, we can always identify the Cauchy problem for which it is a solution. This yields the following lemma:

Lemma A.6.1. *Considering the model (2.7.2) and T_β the GA-revamped time corresponding to t. If there is a strongly continuous semigroup (in T_β), say $\{G_A(T_\beta)\}_{T_\beta\geq 0}$ generated by the operator A then, the family $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$ such that $S_\beta(t) = G(T_\beta)$, is a strongly continuous two-parameter solution operator for the system (2.7.2).*

Theorem A.6.2. *Considering the models (2.7.2) with two different orders β and δ such that $0 < \delta < \beta \leq 1$. Let $\omega \geq 0$ then, $A \in \mathcal{G}^\beta(\omega)$ if and only if $A \in \mathcal{G}^\delta(\omega)$.*

Proof. Suppose $A \in \mathcal{G}^\beta(\omega)$, then, (2.7.2) admits a strongly continuous two-parameter solution operator $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$ satisfying (A.4.6). Hence, by definition we have $\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1} = \{G_A(T_\beta)\}_{T_\beta\geq 0}$ where T_β is GA-revamped time $\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta - \Gamma(\beta)^{-\beta}}{\beta}$, corresponding to t and $\{G_A(T_\beta)\}_{T_\beta\geq 0}$ is a strongly continuous semigroup (in T_β) generated by the operator A. Moreover, by Remark A.4.7, we have $G_A(T_\beta)$ satisfying (A.4.8). For $0 < \delta < \beta \leq 1$, let us define $T_\delta = T_\delta(t) = \frac{\left(t+\frac{1}{\Gamma(\delta)}\right)^\delta - \Gamma(\delta)^{-\delta}}{\delta}$, the GA-revamped time (of order δ) corresponding to t, then $\{G_A(T_\delta)\}_{T_\delta\geq 0}$ is also a strongly continuous semigroup (in T_δ) generated by the operator A since $\{G_A(T_\beta)\}_{T_\beta\geq 0}$ is. Moreover, by (A.4.8) we have

$$\|G_A(T_\delta)\|_{\mathcal{X}} \leq M e^{\omega T_\delta}, \tag{A.6.1}$$

and Lemma A.6.1 concludes the first part of the proof, showing the subordination principle for the model (2.7.2).

Conversely, to prove the prolongation principle, we suppose $A \in \mathcal{G}^\delta(\omega)$ and the rest of the proof follows the same steps as above. ■

The following corollary appears as an immediate consequence.

Corollary A.6.3. *Consider any $\beta \in (0, 1)$. Then, there are constants $\omega \geq 0$ and*

$M \geq 1$ such that the operator A in model (2.7.2) is the infinitesimal generator of a C_0-semigroup $G(t)$ satisfying $\|G(t)\| \leq Me^{\omega t}$, $t \geq 0$ if and only if $A \in \mathcal{G}^{\beta}(M, \omega)$ with the corresponding two-parameter solution operator $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$ satisfying (A.4.6).

A.7 Applications to break-up dynamics in transport-convection

Mathematical settings and Model's analysis

In this section we address the well-posedness of the model

$$ {}_0^A D_t^\beta p(t,x,n) = -div(\omega(x,n)p(t,x,n)) - a_n p(t,x,n) + \sum_{m=n+1}^{\infty} b_{n,m} a_m p(t,x,m), \quad \text{(A.7.1)} $$

where $t > 0,\ 0 < \beta \leq 1,\ x \in \mathbb{R}^3,\ \ n = 1, 2, 3, ...$ and subject to initial conditions

$$ p(0,x,n) = \overset{o}{p}_n(x), \qquad n = 1, 2, 3, ... \quad \text{(A.7.2)} $$

by using the concepts defined here above and setting other suitable conditions. Equation (A.7.1) models the break-up dynamics of moving groups. In terms of the mass size m and the position x, the state of the system is characterized at any moment t by the particle-mass-position distribution $p = p(t, x, m)$, (p is also called the *density* or *concentration* of particles), with $p : \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}_+$, the velocity $\omega = \omega(x, m)$ of the transport is supposed to be a known quantity depending on m and x. The average fragmentation rate a_n is the average number at which clusters of size n undergo splitting, $b_{n,m} \geq 0$ is the average number of n-groups produced upon the splitting of m-groups. The space variable x is supposed to vary in the whole of $\mathbb{R}^3 = \Omega$. The function $\overset{o}{p}_n$ represents the density of n-groups at the beginning of observation ($t = 0$) and it is integrable with respect to x over the full space $\mathbb{R}^3$. The necessary assumptions that will be useful in the analysis are introduced in the following sections.

A.7.1 Well-posedness for the break-up part of the model

Since a group of size $m \leq n$ cannot split to form a group of size n, we require $b_{n,m} = 0$ for all $m \leq n$ and

$$ a_1 = 0, \ \sum_{m=1}^{n-1} m b_{m,n} = n, \quad (n = 2, 3, ...), \quad \text{(A.7.3)} $$

meaning that a cluster of size one cannot split and the sum of all individuals obtained by break-up of an n-group is equal to n. Because the total number of individuals in a population is not modified by interactions among groups and that the mass is expected to be a conserved quantity, the most appropriate Banach space to work in is the space

$$\mathcal{X}_1 := \left\{ \mathbf{g} = (g_n)_{n=1}^{\infty} : \mathbb{R}^3 \times \mathbb{N} \ni (x,n) \to g_n(x), \|\mathbf{g}\|_1 := \int_{\mathbb{R}^3} \sum_{n=1}^{\infty} n|g_n(x)|dx < \infty \right\}. \tag{A.7.4}$$

We work in this space because they have many desirable properties, like controlling the norm of their elements which, in our case, represents the total mass (or total number of individuals) of the system and must be finite. Because uniqueness of solutions to the systems of type (A.7.1)-(A.7.2) is proved to be a more difficult problem [30], we restrict our analysis to a smaller class of functions, so we introduce the following class of Banach spaces (of distributions with finite higher moments)

$$\mathcal{X}_r := \left\{ \mathbf{g} = (g_n)_{n=1}^{\infty} : \mathbb{R}^3 \times \mathbb{N} \ni (x,n) \to g_n(x), \|\mathbf{g}\|_r := \int_{\mathbb{R}^3} \sum_{n=1}^{\infty} n^r|g_n(x)|dx < \infty \right\}, \tag{A.7.5}$$

$r \geq 1$, which coincides with $\mathcal{X}_1$ for $r = 1$. We assume that for each $t \geq 0$, the function $(x,n) \longrightarrow p(t,x,n) = p_n(t,x)$ is such that $\mathbf{p} = (p_n(t,x))_{n=1}^{\infty}$ is from the space $\mathcal{X}_r$ with $r \geq 1$. In $\mathcal{X}_r$ we can rewrite (A.7.1)-(A.7.2) in more compact form,

$$\begin{aligned} {}_0^A D_t^{\beta} \mathbf{p} &:= \mathbf{D}\mathbf{p} + \mathbf{F}\mathbf{p}, \\ \mathbf{p}_{|t=0} &= \mathring{\mathbf{p}}, \end{aligned} \tag{A.7.6}$$

where $t > 0,\ 0 < \beta \leq 1,\ x \in \mathbb{R}^3,\ \ n = 1, 2, 3, \ldots$ Here $\mathbf{p}$ is the vector $(p(t,x,n))_{n\in\mathbb{N}}$, $\mathbf{D}$ the transport expression defined as

$$(p(t,x,n))_{n\in\mathbb{N}} \longrightarrow \left(-\mathrm{div}(\omega(x,n)p(t,x,n))\right)_{n=1}^{\infty}, \tag{A.7.7}$$

$\mathring{\mathbf{p}}$ the initial vector $(\mathring{p}_n(x))_{n\in\mathbb{N}}$ which belongs to $\mathcal{X}_r$ and $\mathbf{F}$ the fragmentation expression defined by

$$(\mathbf{F}\mathbf{p})_{n=1}^{\infty} := \left(-a_n p(t,x,n) + \sum_{m=n+1}^{\infty} b_{n,m} a_m p(t,x,m) \right)_{n=1}^{\infty}.$$

In this work, for any subspace $S \subseteq \mathcal{X}_r$, we will denote by S_+ the subset of S defined as $S_+ = \{\mathbf{g} = (g_n)_{n=1}^{\infty} \in S; g_n(x) \geq 0, n \in \mathbb{N}, x \in \mathbb{R}^3\}$. Note that any $\mathbf{g} \in (\mathcal{X}_r)_+$ possesses moments

$$M_q(\mathbf{g}) := \sum_{n=1}^{\infty} n^q g_n$$

of all orders $q \in [0, r]$. In $\mathcal{X}_r$, we define the operators $\mathbf{A}$ and $\mathbf{B}$ by

$$\mathbf{A}\mathbf{g} := (a_n g_n)_{n=1}^{\infty}, \quad D(\mathbf{A}) := \{\mathbf{g} \in \mathcal{X}_r : \int_{\mathbb{R}^3} \sum_{n=1}^{\infty} n^r a_n |g_n(x)| dx < \infty\}; \tag{A.7.8}$$

Throughout, we assume that the coefficients a_n and $b_{n,m}$ satisfy the mass conservation conditions (A.7.3). Now let's prove that $\mathbf{B}$ is well defined on $D(\mathbf{A})$ as stated in (A.7.10). Making use of the condition (A.7.3), we have

$$n^r - \sum_{m=1}^{n-1} m^r b_{m,n} \geq n^r - (n-1)^{r-1} \sum_{m=1}^{n-1} m b_{m,n} = n^r - n(n-1)^{r-1} \geq 0.$$

Hence

$$\sum_{m=1}^{n-1} m^r b_{m,n} \leq n^r \tag{A.7.9}$$

for $r \geq 1$, $n \geq 2$. Note that the equality holds for $r = 1$. For every $\mathbf{g} \in D(\mathbf{A})$, we have then

$$\begin{aligned}
\|\mathbf{Bg}\|_r &= \int_{\mathbb{R}^3} \sum_{n=1}^{\infty} n^r \left(\sum_{m=n+1}^{\infty} b_{n,m} a_m |g_m(x)| \right) dx \\
&= \int_{\mathbb{R}^3} \sum_{m=2}^{\infty} a_m |g_m(x)| \left(\sum_{n=1}^{\infty} n^r b_{n,m} \right) dx \\
&= \int_{\mathbb{R}^3} \sum_{m=2}^{\infty} a_m |g_m(x)| \left(\sum_{n=1}^{m-1} n^r b_{n,m} \right) dx \\
&\leq \int_{\mathbb{R}^3} \sum_{m=2}^{\infty} a_m |g_m(x)| m^r dx \\
&= \|\mathbf{Ag}\|_r \\
&< \infty,
\end{aligned}$$

$$\mathbf{Bg} := (B_n g_n)_{n=1}^{\infty} = \left(\sum_{m=n+1}^{\infty} b_{n,m} a_m g_m \right)_{n=1}^{\infty}, \quad D(\mathbf{B}) := D(\mathbf{A}). \tag{A.7.10}$$

where we have used the inequality (A.7.9). Then $\|\mathbf{Bg}\|_r \leq \|\mathbf{Ag}\|_r$, for all $\mathbf{g} \in D(\mathbf{A})$, so that we can take $D(\mathbf{B}) := D(\mathbf{A})$ and $(\mathbf{A} + \mathbf{B},\, D(\mathbf{A}))$ is well-defined.

A.7.2 Well-posedness for the transport part of the model

Our primary objective in this section is to analyze the solvability of the Cauchy problem for the transport equation

$${}_0^A D_t^{\beta} p(t,x,n) = -div(\omega(x,n)\, p(t,x,n)), \tag{A.7.11}$$

$$p(0,x,n) = \overset{o}{p}_n(x), \qquad n = 1, 2, 3, ...$$

in the space $\mathcal{X}_r$, where $t > 0$, $0 < \beta \leq 1$, $x \in \mathbb{R}^3$, $n = 1, 2, 3, ...$
To do so we need the following:

Now let us fix $n \in \mathbb{N}$. We consider the function $\omega_n : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by $\omega_n(x) = \omega(x,n)$ and $\tilde{\mathcal{D}}_n$ the expression appearing on the right-hand side of the equation (A.7.11). Then

$$\begin{aligned}\tilde{\mathcal{D}}_n[p(t,x,n)] &:= -div\,(\omega(x,n)\,p(t,x,n))\\ &= (\nabla\cdot\omega(x,n))p(t,x,n) + \omega(x,n)\cdot(\nabla p(t,x,n)).\end{aligned} \tag{A.7.12}$$

We assume that ω_n is divergence free and globally Lipschitz continuous. Then $div\,\omega_n(x) := \nabla\cdot\omega(x,n) = 0$ and (A.7.12) becomes

$$\tilde{\mathcal{D}}_n[p(t,x,n)] := \omega(x,n)\cdot(\nabla p(t,x,n)). \tag{A.7.13}$$

We note that the operators on the right-hand side of (A.7.6) have the property that one of the variables is a parameter and, for each value of this parameter, the operator has a certain desirable property (like being the generator of a semigroup) with respect to the other variable. Thus we need to work with parameter-dependent operators that can be "glued" together in such a way that the resulting operator inherits the properties of the individual components. Let us provide a framework for such a technique called the method of semigroups with a parameter [30]. Let us consider the space $\mathcal{X} := L_p(S,X)$ where $1 \le p < \infty$, (S,m) is a measure space and X a Banach space. Let us suppose that we are given a family of operators $\{(A_s, D(A_s))\}_{s\in S}$ in X and define the operator $(\mathbb{A}, D(\mathbb{A}))$ acting in $\mathcal{X}$ according to the following formulae,

$$\mathcal{D}(\mathbb{A}) := \{g \in \mathcal{X}; g(s) \in D(A_s) \text{ for almost every } s \in S,\ \ \mathbb{A}g \in \mathcal{X}\}, \tag{A.7.14}$$

and, for $g \in \mathcal{D}(\mathbb{A})$,

$$(\mathbb{A}g)(s) := A_s g(s), \tag{A.7.15}$$

for every $s \in S$.
We set

$$X_x := L_1(\mathbb{R}^3, dx) := \{\psi : \|\psi\| = \int\limits_{\mathbb{R}^3} |\psi(x)|dx < \infty\}$$

and define in X_x the operators $(\mathcal{D}_n, D(\mathcal{D}_n))$ as

$$\begin{aligned}\mathcal{D}_n p_n &= \tilde{\mathcal{D}}_n p_n, \quad \text{with } \tilde{\mathcal{D}}_n p_n \text{ represented by } (A.7.13)\\ D(\mathcal{D}_n) &:= \{p_n \in X_x,\ \ \mathcal{D}_n p_n \in X_x\}, \quad n \in \mathbb{N}.\end{aligned} \tag{A.7.16}$$

Then, in $\mathcal{X}_r$ we can define for the operator $\mathbf{D}$ (A.7.7) the domain

$$D(\mathbf{D}) = \{\mathbf{p} = (p_n)_{n\in\mathbb{N}} \in \mathcal{X}_r,\ p_n \in D(\mathcal{D}_n) \text{ for almost every } n \in \mathbb{N},\ \mathbf{D}\mathbf{p} \in \mathcal{X}_r\}. \tag{A.7.17}$$

Theorem A.7.1. *Let us fix any $\beta \in (0,1]$. If for each $n \in \mathbb{N}$ the function $\omega_n : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is globally Lipschitz continuous and divergence-free then, the operator $(D(\boldsymbol{D}), \boldsymbol{D})$ is the generator of a contractive strongly continuous two-parameter solution operator for the system (A.7.11).*

Proof. To prove it we apply the subordination principle of Theorem A.6.2, by considering the model (A.7.11) with $\beta = 1$ to have the compact form

$$\partial_t \mathbf{P} = \mathbf{D}\mathbf{P}, \tag{A.7.18}$$

subject to the initial condition

$$\mathbf{p}|_{t=0} = \overset{\circ}{\mathbf{p}}. \tag{A.7.19}$$

where $\mathbf{D}$ the transport expression defined in (A.7.7). Making use of [30, Theorem 2], it is proved that if the conditions of Theorem A.7.1 are satisfied then, there exists a strongly continuous stochastic (positive and contractive) semigroup generated by $(D(\mathbf{D}), \mathbf{D})$. Hence, $\mathbf{D} \in \mathcal{G}^1(1, 0)$ and exploiting the the subordination principle of Theorem A.6.2, we have $\mathbf{D} \in \mathcal{G}^\beta(1, 0)$, which prove the theorem. ■

A.7.3 Existence results for the full model

Attention is now shifted to the transport problem with the loss part of the break-up process. We assume that there are two constants $0 < \theta_1$ and θ_2 such that for every $x \in \mathbb{R}^3$,

$$\theta_1 \alpha_n \leq a_n(x) \leq \theta_2 \alpha_n, \tag{A.7.20}$$

with $\alpha_n \in \mathbb{R}_+$ and independent of the state variable x. Then a_n is bounded for each $n \in \mathbb{N}$ and the loss operator $(A_n, D(A_n))$ can be defined in X_x as $A_n(x) = a_n(x)$ with $D(A_n) = X_x = L_1(\mathbb{R}^3)$. The corresponding abstract Cauchy problem for the full model (A.7.1)-(A.7.2) reads as

$$\begin{aligned} {}_0^A D_t^\beta \mathbf{p} &= \mathbf{D}\mathbf{p} + \mathbf{F}\mathbf{p} \\ \mathbf{p}_{|_{t=0}} &= \overset{\circ}{\mathbf{p}}. \end{aligned} \tag{A.7.21}$$

The following theorem holds.

Theorem A.7.2. *Assume that (A.7.20) is satisfied for each $n \in \mathbb{N}$.*
There is an extension $(\mathcal{K}, D(\mathcal{K}))$ of $(\boldsymbol{D} + \boldsymbol{F}, D(\boldsymbol{D}) \cap D(\boldsymbol{A}))$ that generates, on $\mathcal{X}_r$, a strongly continuous two-parameter solution operator for the system (A.7.1)-(A.7.2) which is contractive.

Proof. The proof follows the same steps as the proof of Theorem A.7.1 where we apply the subordination principle on the reference [30, Theorem 5]. ■

This concludes, as an application, the well-posedness of an integrodifferential equation modeling convection and break-up processes. It is certain that this whole dissertation will inspire more than one author with the introduction of the new concepts. Thus, it emerges to be a breakthrough that might help solving opens problems mentioned throughout this thesis or lead to more complex analysis of evolutions equations often describing phenomena more and more intricate.